On the Topology of the Reduced Classical Configuration Space of Lattice QCD

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Abstract

We study the topological structure of the quotient $\mathcal{X}$ of $\mathrm{SU}(3)\times\mathrm{SU}(3)$ by diagonal conjugation. This is the simplest nontrivial example for the classical reduced configuration space of chromodynamics on a spatial lattice in the Hamiltonian approach. We construct a cell complex structure of $\mathcal{X}$ in such a way that the closures of strata are subcomplexes and compute the homology and cohomology groups of the strata and their closures.

1 Introduction

This paper is a continuation of [6] where we have studied the quotient of $\mathrm{SU}(3) \times \mathrm{SU}(3)$ by diagonal conjugation by means of invariant theory. Let us start with some introductory remarks on the motivation for the study of such a very specific space.

Many aspects of the rich mathematical and physical structure of nonabelian gauge theories are not accessible by perturbation theory, but only by nonperturbative methods. Perhaps the most prominent of these aspects is the low energy hadron physics and, in particular, quark confinement. Another such aspect is the stratified structure of the classical configuration space of the theory, i.e., of the space of orbits of the group of local gauge transformations acting on the classical fields. This space consists of an open dense manifold part, the 'principal stratum', and several singularities which themselves decompose into manifolds of varying dimension, the 'non-principal' or 'secondary strata'. We note that similar structures arise in the study of the geometry of quantum mechanical state spaces [1, 3]. While, by now, much is known about the stratified structure itself, see [17, 18], it is still open whether and how it expresses itself in the physical properties of the theory. A systematic investigation of this question requires a concept of how to encode the stratification into the quantum theory. A promising candidate is the concept of costratified Hilbert space recently developed by Huebschmann [12]. Very roughly, a costratified Hilbert space consists of a total Hilbert space $\mathcal{H}$ and a family of Hilbert spaces $\mathcal{H}_i$ associated with the closures of the strata, together with a family of bounded linear maps $\mathcal{H}_i \to \mathcal{H}_j$ whenever stratum j is contained in the closure of stratum i. As was demonstrated in [12], one way to construct such costratified Hilbert space is by extending methods of geometric quantization to so-called stratified symplectic spaces. (However, the concept of costratified Hilbert space does not rely on any particular quantization procedure.) A stratified symplectic space is a stratified space in the usual sense where, in addition, the total space carries a Poisson structure, the strata carry symplectic structures and the structures are compatible in the sense that the injections of the strata into the total space are Poisson maps. Such spaces naturally emerge as the reduced phase spaces of Hamiltonian systems with symmetries by the process of singular Marsden-Weinstein reduction [7]. These observations suggest the following strategy to construct a quantum gauge theory with the stratification encoded [20]: formulate the classical theory as a Hamiltonian system with symmetries, construct the reduced phase space and then try to apply the method of stratified Kähler quantization developed in [12] to obtain a costratified Hilbert space. To separate the difficulties arising from infinite dimensions from those related to symmetry reduction it is reasonable to first work in the lattice approximation. For a simple model motivated by SU(2)-lattice gauge theory the program was carried out in [13].

For the construction explained above, and of course for the discussion of classical and quantum dynamics anyhow, it is important to know the topological and geometric structure of the reduced phase space as well as of the reduced configuration space. In this work, we focus on the latter. Using a tree gauge, it can be shown to be given by the quotient of several copies of the (compact) gauge group by diagonal conjugation. Since we are interested in QCD, our gauge group is SU(3). Since for one copy of SU(3) the quotient is

well known to be a Weyl alcove in the subgroup of diagonal matrices, the simplest nontrivial case is 2 copies. This corresponds to a lattice consisting of 2 plaquettes. The description of the quotient of 2 copies by means of invariant theory was derived in a previous paper [6]. In the present work we construct a cell complex structure which is compatible with the orbit type stratification and use it to compute the homology and cohomology groups of the quotient space as well as of the strata and their closures.

To conclude these introductory notes, let us remark the following. The strategy explained above follows the path to first reduce the symmetries and then quantize. One can as well follow the alternative path of reducing the symmetries on the quantum level, i.e., to start from a field algebra and to construct the algebra of observables by implementing gauge invariance and the local Gauss law, see [15]. To implement the stratification in this approach, one has to work out the concept of (co)stratified observable algebra and to modify the construction in [15] so as to provide, in addition, observable algebras associated with the strata. In a second step, one has to study representations of the costratified observable algebra so obtained. Of course, such representations act in a costratified Hilbert space. Also for this step one should start from the results in [15] and try to extend the methods used there to the costratified case. Although this alternative strategy seems equally promising to us, no work in this direction has been done yet.

The paper is organized as follows. In Section 2 we define the space we will study and introduce some notation. In Section 3 we relate the reduced configuration space to certain double quotients of U(3) and show how to construct a cell complex structure of the reduced configuration space from cell decompositions of the double quotients. Section 4 is devoted to the study of the most important of these double quotients, i.e., $T\backslash \mathrm{U}(3)/T$, where T denotes the subgroup of U(3) of diagonal matrices. The remaining double quotients and the factorization maps between them are studied in Section 5. Based on these preparations, in Section 6 the desired cell complex structure of the reduced configuration space is given and the boundary operator is computed. In Section 7 it is shown that the closures of the strata are subcomplexes. Finally, in Section 8 the homology and cohomology groups of the reduced configuration space and its strata are derived. We conclude with a brief discussion.

2 Basic notation

Let $\mathcal{X}$ denote the quotient space of the action of SU(3) on SU(3) × SU(3) by diagonal conjugation,

$$(a,b) \mapsto (gag^{-1}, gbg^{-1}) , \qquad a,b,g \in \mathrm{SU}(3) .$$

This space can be interpreted as the reduced configuration space of an SU(3)-lattice gauge model defined on a lattice consisting of two plaquettes. See [6] for details. The space $\mathcal{X}$ is stratified by the orbit types of the action. The stratification will be explained in detail in Section 7. The aim of this paper is to construct a cell decomposition of $\mathcal{X}$ such that the closures of the strata are subcomplexes and to use it to compute the homology and cohomology groups of the strata and their closures.

Let us introduce some notation. Consider the toral subgroup of diagonal matrices a in SU(3). The conditions $a_{11} = a_{22}$, $a_{11} = a_{33}$, $a_{22} = a_{33}$ on the entries of a define one-parameter subgroups that cut this toral subgroup into 6 closed triangular subsets. Let $\mathcal{A}$ denote one of them. The embedding $\mathcal{A} \to \mathrm{SU}(3)$ descends to a homeomorphism of $\mathcal{A}$ onto the quotient of SU(3) by inner automorphisms (see [6] for more details). $\mathcal{A}$ has a natural cell decomposition consisting of a 2-cell $\mathcal{A}^2 \equiv \mathcal{A}$, three 1-cells ('edges') $\mathcal{A}_i^1$ and three 0-cells ('vertices') $\mathcal{A}_i^0$ which make up the center $\mathbb{Z}_3$ of SU(3). We assume the 1-cells to be oriented by the boundary orientation induced from $\mathcal{A}^2$. In the sequel, for the cells of $\mathcal{A}$ we will loosely write $\mathcal{A}_i^p$, although for $p = 2$ the index i is redundant. The interior of $\mathcal{A}_i^p$ will be denoted by $\dot{\mathcal{A}}_i^p$.

We will also work with U(3) and its subgroup of diagonal matrices $T \cong \mathrm{U}(1)^3$, as well as the subgroups $U_1, U_2, U_3 \cong \mathrm{U}(1) \times \mathrm{U}(2)$, consisting of the matrices

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{pmatrix}, \qquad \begin{pmatrix} b_{11} & 0 & b_{12} \\ 0 & \alpha & 0 \\ b_{21} & 0 & b_{22} \end{pmatrix}, \qquad \begin{pmatrix} b_{11} & b_{12} & 0 \\ b_{21} & b_{22} & 0 \\ 0 & 0 & \alpha \end{pmatrix},$$

respectively, where $\alpha \in \mathrm{U}(1)$ and $b \in \mathrm{U}(2)$.

Next, for $i = 1, 2, 3$, let $i_\pm := i \pm 1 \mod 3$. The elements of the permutation group $\mathcal{S}_3$ are denoted by τ or σ. We number them in the following way: τ_0 denotes the identity permutation, τ_i for $i = 1, 2, 3$ denotes the transposition of i_+ and i_- and τ_4 and τ_5 are the backward and forward cyclic permutations, respectively. A representation of $\mathcal{S}_3$ in U(3) is given by the matrices

$$\tau_{ij} = \delta_{\tau(i)\,j}\,, \qquad \tau \in \mathcal{S}_3\,,$$

where δ_{kl} denotes the Kronecker symbol. Explicitly,

$$\tau_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The centralizer and the normalizer of a subset $A \subseteq G$ will be denoted by $\mathrm{C}_G(A)$ and $\mathrm{N}_G(A)$, respectively.

3 The configuration space in terms of double quotients

The starting point of our considerations is the map

$$\varphi : \mathcal{A} \times \mathcal{A} \times \mathrm{U}(3) \to \mathcal{X}\,, \qquad (t, s, g) \mapsto [(t, gsg^{-1})]\,,$$

where $[\,\cdot\,]$ denotes the class w.r.t. the diagonal action of SU(3).

Lemma 3.1. *The map φ is surjective and closed. Equality $\varphi(t, s, g) = \varphi(t', s', g')$ holds iff $t = t'$, $s = s'$ and $g' = hgk$ for some $h \in \mathrm{C}_{\mathrm{U}(3)}(t)$ and $k \in \mathrm{C}_{\mathrm{U}(3)}(s)$.*

Proof. φ is surjective: Let $(a,b) \in \mathrm{SU}(3) \times \mathrm{SU}(3)$. There exist $c,d \in \mathrm{SU}(3)$ such that $t := cac^{-1}$ and $s := dbd^{-1}$ are in $\mathcal{A}$. Denote $g := cd^{-1}$. Then $(t, gsg^{-1}) = (cac^{-1}, cbc^{-1})$, hence $\varphi(t,s,g) = [(a,b)]$. φ is closed, because it is a map from a compact space to a Hausdorff space. To determine the preimages, let (t,s,g) and (t',s',g') be given. If t', s' and g' are as in the lemma then they are obviously mapped by φ to the same point in $\mathcal{X}$. Conversely, assume that $\varphi(t',s',g') = \varphi(t,s,g)$. Then there exists $h \in \mathrm{SU}(3)$ such that $t' = hth^{-1}$ and $g's'g'^{-1} = h(gsg^{-1})h^{-1}$. Since t' and t are both in $\mathcal{A}$, the first equality implies $t' = t$, hence $h \in C_{\mathrm{SU}(3)}(t) \subseteq C_{\mathrm{U}(3)}(t)$. Similarly, the second equality implies $g'^{-1}hg \in C_{\mathrm{U}(3)}(s)$. Denoting $k := (g'^{-1}hg)^{-1}$ we obtain $g' = hgk$, as asserted. $\square$

Thus, up to diagonal conjugacy, pairs $(a,b) \in \mathrm{SU}(3)\times\mathrm{SU}(3)$ can be characterized by triples (t,s,g) with s and t being uniquely determined and g representing a class in the double quotient $\mathrm{C}_{\mathrm{U}(3)}(t)\backslash \mathrm{U}(3)/\mathrm{C}_{\mathrm{U}(3)}(s)$, taken wrt. left and right multiplication, respectively. Since for any t in the interior of the cell $\mathcal{A}_i^p$ of $\mathcal{A}$, the centralizer is $C_{\mathrm{U}(3)}(t) = C_{\mathrm{U}(3)}(\mathcal{A}_i^p)$, to any pair of cells $\mathcal{A}_i^p$, $\mathcal{A}_j^q$ there corresponds a double quotient

$$D(\mathcal{A}_i^p, \mathcal{A}_j^q) := \mathrm{C}_{\mathrm{U}(3)}(\mathcal{A}_i^p)\backslash \mathrm{U}(3)/\mathrm{C}_{\mathrm{U}(3)}(\mathcal{A}_j^q)\,.$$

In detail, we find

$$\begin{aligned}
&D(\mathcal{A}^2,\mathcal{A}^2) = T\backslash \mathrm{U}(3)/T\,, \quad D(\mathcal{A}_i^1,\mathcal{A}^2) = U_i\backslash \mathrm{U}(3)/T\,, \quad D(\mathcal{A}^2,\mathcal{A}_i^1) = T\backslash \mathrm{U}(3)/U_i\,,\\
&D(\mathcal{A}_i^1,\mathcal{A}_j^1) = U_i\backslash \mathrm{U}(3)/U_j\,,\\
&D(\mathcal{A}_i^0,\mathcal{A}^2) = D(\mathcal{A}^2,\mathcal{A}_i^0) = D(\mathcal{A}_i^0,\mathcal{A}_j^1) = D(\mathcal{A}_i^1,\mathcal{A}_j^0) = D(\mathcal{A}_i^0,\mathcal{A}_j^0) = \{*\}\,.
\end{aligned} \tag{3.1}$$

There exist the following natural factorization maps:

$$\begin{tikzcd}
T\backslash \mathrm{U}(3)/T \arrow[r, "\lambda_i^{12}"] \arrow[d, "\lambda_j^{21}"'] & U_i\backslash \mathrm{U}(3)/T \arrow[d, "\mu_{ij}^{12}"] & \\
T\backslash \mathrm{U}(3)/U_j \arrow[r, "\mu_{ij}^{21}"] & U_i\backslash \mathrm{U}(3)/U_j \arrow[r] & \{*\}
\end{tikzcd} \tag{3.2}$$

The idea of the construction of the cell decomposition of $\mathcal{X}$ can be stated as follows. According to Lemma 3.1, φ induces maps

$$\pi_{ij}^{pq} : \mathcal{A}_i^p \times \mathcal{A}_j^q \times D(\mathcal{A}_i^p, \mathcal{A}_j^q) \to \mathcal{X} \tag{3.3}$$

which, since φ is closed, restrict to homeomorphisms of $\dot{\mathcal{A}}_i^p \times \dot{\mathcal{A}}_j^q \times D(\mathcal{A}_i^p, \mathcal{A}_j^q)$ onto the image of this subset in $\mathcal{X}$. Hence, if one has a cell K in the double quotient $D(\mathcal{A}_i^p, \mathcal{A}_j^q)$ then $\mathcal{A}_i^p \times \mathcal{A}_j^q \times K$ with characteristic map obtained by restriction of π_{ij}^{pq} is a candidate for a cell of $\mathcal{X}$. The potential identifications on the boundary can be kept track of by exploring, for each boundary cell $\mathcal{A}_{i'}^{p\text{-}1}$ of $\mathcal{A}_i^p$ and $\mathcal{A}_{j'}^{q\text{-}1}$ of $\mathcal{A}_j^q$, the relation between π_{ij}^{pq} and the maps $\pi_{i'j}^{p\text{-}1\;q}$ and $\pi_{ij'}^{p\;q\text{-}1}$, respectively. More generally, the situation is the following. Given $\mathcal{A}_{i'}^{p'}$ and $\mathcal{A}_{j'}^{q'}$ such that

$$\mathcal{A}_i^p \subseteq \mathcal{A}_{i'}^{p'}\,, \qquad \mathcal{A}_j^q \subseteq \mathcal{A}_{j'}^{q'}\,, \tag{3.4}$$

the factorization map $D(\mathcal{A}^{p'}_{i'},\mathcal{A}^{q'}_{j'}) \to D(\mathcal{A}^p_i,\mathcal{A}^q_j)$ exists–it is in fact a composition of some of the maps (3.2)–and the diagram

$$\begin{tikzcd} \mathcal{A}^p_i \times \mathcal{A}^q_j \times D(\mathcal{A}^{p'}_{i'},\mathcal{A}^{q'}_{j'}) \arrow[r] \arrow[d] & \mathcal{A}^{p'}_{i'} \times \mathcal{A}^{q'}_{j'} \times D(\mathcal{A}^{p'}_{i'},\mathcal{A}^{q'}_{j'}) \arrow[d, "\pi^{p'q'}_{i'j'}"] \\ \mathcal{A}^p_i \times \mathcal{A}^q_j \times D(\mathcal{A}^p_i,\mathcal{A}^q_j) \arrow[r, "\pi^{pq}_{ij}"] & \mathcal{X} \end{tikzcd} \tag{3.5}$$

commutes, where the upper horizontal arrow is given by the natural injection and the left vertical arrow is given by the identical map of $\mathcal{A}^p_i \times \mathcal{A}^q_j$ times the above factorization map. Thus, in effect it is this factorization map which carries the information about the boundary identifications carried out by the (prospective) characteristic map.

Theorem 3.2. *Let there be given cell decompositions of the double quotients* (3.1) *such that the factorization maps* (3.2) *are cellular. Then the collection of all products $\mathcal{A}^p_i \times \mathcal{A}^q_j \times K$, where $\mathcal{A}^p_i$, $\mathcal{A}^q_j$ are cells of $\mathcal{A}$, K is a cell of $D(\mathcal{A}^p_i,\mathcal{A}^q_j)$ and the characteristic map is defined by restriction of π^{pq}_{ij}, defines a cell complex structure on $\mathcal{X}$. If the cells are oriented by the natural product orientation, the boundary operator is given by*

$$\partial(\mathcal{A}^p_i \times \mathcal{A}^q_j \times K) = \sum\nolimits_{i'} \mathcal{A}^{p\text{-}1}_{i'} \times \mathcal{A}^q_j \times \rho_{i'*}(K) + (-1)^p \sum\nolimits_{j'} \mathcal{A}^p_i \times \mathcal{A}^{q\text{-}1}_{j'} \times \sigma_{j'*}(K) \\ + (-1)^{p+q}\mathcal{A}^p_i \times \mathcal{A}^q_j \times \partial K\,,$$

where the sums run over the boundary cells of $\mathcal{A}^p_i$ and $\mathcal{A}^q_j$, respectively, equipped with the correct sign, and $\rho_{i'}$ and $\sigma_{j'}$ stand for the factorization maps $D(\mathcal{A}^p_i,\mathcal{A}^q_j) \to D(\mathcal{A}^{p\text{-}1}_{i'},\mathcal{A}^q_j)$ and $D(\mathcal{A}^p_i,\mathcal{A}^q_j) \to D(\mathcal{A}^p_i,\mathcal{A}^{q\text{-}1}_{j'})$, respectively, given in (3.2).

Proof. We give an inductive construction of the skeleta. For the sets $\mathcal{A}^p_i \times \mathcal{A}^q_j \times K$ we will use the shorthand notation C^{pqr}_{ijk}, where r stands for the dimension of K and k is a virtual label for the r-cells of $D(\mathcal{A}^p_i,\mathcal{A}^q_j)$. Moreover, in order to distinguish between the boundary of C^{pqr}_{ijk} in $\mathcal{X}$ and its boundary in the cell complex $\mathcal{A}^p_i \times \mathcal{A}^q_j \times D(\mathcal{A}^p_i,\mathcal{A}^q_j)$, the first one will be denoted by ∂ and the second one by $\tilde\partial$. In a sense, $\tilde\partial C^{pqr}_{ijk}$ is the 'natural' boundary of C^{pqr}_{ijk}.
We will say that a cell complex $\mathcal{X}^n$ has the property $(*)$ iff it is homeomorphic to the image of the n-skeleton of $\mathcal{A}^2 \times \mathcal{A}^2 \times D(\mathcal{A}^2,\mathcal{A}^2)$ under π^{22} (and can thus be identified with a subset of $\mathcal{X}$).
We start with defining $\mathcal{X}^0$ to consist of the nine isolated points $C^{000}_{ij} = \mathcal{A}^0_i \times \mathcal{A}^0_j \times \{*\}$, $i,j = 1,2,3$. Then $\mathcal{X}^0$ is a cell complex and $(*)$ holds trivially.
Now assume that the C^{pqr}_{ijk} with $p+q+r=n$ constitute a cell complex $\mathcal{X}^n$ which has the property $(*)$ and let some C^{pqr}_{ijk} with $p+q+r=n+1$ be given. Consider the diagram (3.5) with $p'=q'=2$. Due to the assumption that the factorization maps (3.2) are cellular, the left vertical arrow in this diagram is cellular. Hence, the preimage of $\tilde\partial C^{pqr}_{ijk}$ under this map is a union of n-cells of $\mathcal{A}^2 \times \mathcal{A}^2 \times D(\mathcal{A}^2,\mathcal{A}^2)$ and is therefore mapped by π^{22} to $\mathcal{X}^n$, due to $(*)$. Then the diagram yields $\pi^{pq}_{ij}(\tilde\partial C^{pqr}_{ijk}) \subseteq \mathcal{X}^n$.

Thus, $\mathcal{X}^n$, together with the C_{ijk}^{pqr} of $p+q+r=n+1$ and the corresponding restrictions of the π_{ij}^{pq} taken as characteristic maps, define a cell complex $\mathcal{X}^{n+1}$. We have to show that $\mathcal{X}^{n+1}$ has the property $(*)$. For that purpose, let $\mathcal{X}_0^{n+1}$ denote the topological direct sum of $\mathcal{X}^n$ with all the cells C_{ijk}^{pqr} of dimension $p+q+r=n+1$ and let f denote the union of the attaching maps of these cells. By definition, $\mathcal{X}^{n+1}$ is the quotient of $\mathcal{X}_0^{n+1}$ by the equivalence relation

$$x_1 \sim_f x_2 \quad \Leftrightarrow \quad x_1 = x_2 \text{ or } x_1 = f(x_2) \text{ or } f(x_1) = x_2 \text{ or } f(x_1) = f(x_2)\,,$$

whenever $f(x_1)$ or $f(x_2)$ are defined. Define a map $\psi : \mathcal{X}_0^{n+1} \to \mathcal{X}$ to be the identity on $\mathcal{X}^n$ and π_{ij}^{pq} on C_{ijk}^{pqr}. Since the domain is compact and the target space is Hausdorff, ψ is closed, and hence induces a homeomorphism between its image and the quotient of $\mathcal{X}_0^{n+1}$ obtained by contraction of preimages. The image of ψ can be easily found to be the image of the $(n+1)$-skeleton of $\mathcal{A}^2 \times \mathcal{A}^2 \times D(\mathcal{A}^2, \mathcal{A}^2)$ under π^{22}. Thus, all we have to check is that $x_1 \sim_f x_2$ iff $\psi(x_1) = \psi(x_2)$.
First, assume that $x_1 \sim_f x_2$. If $x_1 = x_2$ then trivially $\psi(x_1) = \psi(x_2)$. If $x_1 = f(x_2)$ then $x_1 \in \mathcal{X}^n$ and x_2 belongs to the boundary of one of the C_{ijk}^{pqr}. Then $x_1 = \psi(x_1)$ and $f(x_2) = \pi_{ij}^{pq}(x_2) = \psi(x_2)$, hence $\psi(x_1) = \psi(x_2)$, too. A similar argument applies to the cases $f(x_1) = x_2$ and $f(x_1) = f(x_2)$.
For the converse implication, we need the following lemma.

Lemma 3.3. *Let $x \in C_{ijk}^{pqr}$ and assume that there exists $x' \in C_{i'j'k'}^{p'q'r'}$, where $p+q+r \geq p'+q'+r'$ and $x' \neq x$, such that $\pi_{ij}^{pq}(x) = \pi_{i'j'}^{p'q'}(x')$. Then $x \in \partial C_{ijk}^{pqr}$.*

According to Lemma 3.1, $\pi_{ij}^{pq}(x) = \pi_{i'j'}^{p'q'}(x')$ implies that x and x' have the same $\mathcal{A}$-parts, i.e., $x = (a, b, y)$ and $x' = (a, b, y')$, where $y \in D(\mathcal{A}_i^p, \mathcal{A}_j^q)$ and $y' \in D(\mathcal{A}_{i'}^{p'}, \mathcal{A}_{j'}^{q'})$. Assume that x is in the interior of C_{ijk}^{pqr}. Then a and b are in the interiors of $\mathcal{A}_i^p$ and $\mathcal{A}_j^q$ which are therefore intersected by $\mathcal{A}_{i'}^{p'}$ and $\mathcal{A}_{j'}^{q'}$, respectively. Hence, (3.4) holds and we have the commutative diagram (3.5), where the left vertical arrow is again cellular by assumption. Hence, the image $\tilde{y}$ of y' under this map belongs to an s-cell of $D(\mathcal{A}_i^p, \mathcal{A}_j^q)$, where $s \leq r'$. Due to the diagram, $\pi_{ij}^{pq}(a, b, \tilde{y}) = \pi_{i'j'}^{p'q'}(a, b, y')$ and hence $\pi_{ij}^{pq}(a, b, \tilde{y}) = \pi_{ij}^{pq}(a, b, y)$. Since π_{ij}^{pq} is injective on $\dot{\mathcal{A}}_i^p \times \dot{\mathcal{A}}_j^q \times D(\mathcal{A}_i^p, \mathcal{A}_j^q)$, then $\tilde{y} = y$. We conclude that y belongs to an s-cell of $D(\mathcal{A}_i^p, \mathcal{A}_j^q)$. Since x is in the interior of C_{ijk}^{pqr}, y is in the interior of the corresponding r-cell of $D(\mathcal{A}_i^p, \mathcal{A}_j^q)$, hence $s \geq r$ and so $r' \geq r$. Then, under the assumption $p+q+r \geq p'+q'+r'$, (3.4) implies $p' = p$ and $q' = q$ and, consequently, $i' = i$, $j' = j$. It follows $y' = \tilde{y}$, hence $y' = y$, hence $x = x'$, in contradiction to the assumption. This proves the lemma.

We continue with the proof of Theorem 3.2. Assume that $\psi(x_1) = \psi(x_2)$. If $x_1 = x_2$ then $x_1 \sim_f x_2$. Hence, let us assume $x_1 \neq x_2$. Then x_1 and x_2 cannot both belong to $\mathcal{X}^n$, because ψ is the identity there. If one of them, say x_1, belongs to $\mathcal{X}^n$ and the other one to one of the $n+1$-cells, say $x_2 \in C_{ijk}^{pqr}$, then $\psi(x_2) = x_1$. Of course, x_1 belongs to the interior of some cell $C_{i'j'k'}^{p'q'r'}$ with $p'+q'+r' \leq n$ and we have $x_1 \equiv \pi_{i'j'}^{p'q'}(x_1)$, as we have identified the interior of $C_{i'j'k'}^{p'q'r'}$ through $\pi_{i'j'}^{p'q'}$ with its image in $\mathcal{X}$. Then Lemma 3.3 yields

$x_2 \in \partial C^{pqr}_{ijk}$. It follows $f(x_2) = \pi^{pq}_{ij}(x_2) = \psi(x_2)$, hence $f(x_2) = x_1$, i.e., $x_1 \sim_f x_2$. If x_1 and x_2 both belong to one of the $n+1$-cells C^{pqr}_{ijk}, then Lemma 3.3, applied to both x_1 and x_2, implies that they belong to the respective boundaries. Therefore, as above, $f(x_l) = \psi(x_l)$, $l = 1, 2$. It follows $f(x_1) = f(x_2)$, hence $x_1 \sim_f x_2$, too.

Finally, we determine the boundary operator. Let $\mathcal{A}^p_i \times \mathcal{A}^q_j \times K$ be given. The boundary operator on the level of the cell complex $\mathcal{A}^p_i \times \mathcal{A}^q_j \times D(\mathcal{A}^p_i, \mathcal{A}^q_j)$ is given by

$$\tilde{\partial}(\mathcal{A}^p_i \times \mathcal{A}^q_j \times K) = \partial\mathcal{A}^p_i \times \mathcal{A}^q_j \times K + (-1)^p \mathcal{A}^p_i \times \partial\mathcal{A}^q_j \times K + (-1)^{p+q} \mathcal{A}^p_i \times \mathcal{A}^q_j \times \partial K\,, \quad (3.6)$$

where the boundaries on the rhs. are taken in $\mathcal{A}$ and $D(\mathcal{A}^p_i, \mathcal{A}^q_j)$, respectively. In order to obtain $\partial(\mathcal{A}^p_i \times \mathcal{A}^q_j \times K)$ from this formula we have to replace each cell C appearing on the rhs. by the cells of $\mathcal{X}$ which span $\pi^{pq}_{ij}(C)$. Since the last term already consists of cells of $\mathcal{X}$, it remains unchanged. The first term is a sum over cells of the type $\mathcal{A}^{p\text{-}1}_{i'} \times \mathcal{A}^q_j \times K$, where $\mathcal{A}^{p\text{-}1}_{i'}$ is one of the boundary cells of $\mathcal{A}^p_i$ (equipped with the correct sign). We have

$$\pi^{pq}_{ij}\Big(\mathcal{A}^{p-1}_{i'} \times \mathcal{A}^q_j \times K\Big) = \pi^{p\text{-}1\ q}_{i'j}\Big(\mathcal{A}^{p-1}_{i'} \times \mathcal{A}^q_j \times \rho_{i'}(K)\Big)\,,$$

where $\rho_{i'}$ stands for the factorization map $D(\mathcal{A}^p_i, \mathcal{A}^q_j) \to D(\mathcal{A}^{p\text{-}1}_{i'}, \mathcal{A}^q_j)$. Since the argument of $\pi^{p\text{-}1\ q}_{i'j}$ on the rhs. consists of cells of $\mathcal{X}$, it replaces $\mathcal{A}^{p-1}_{i'} \times \mathcal{A}^q_j \times K$ in (3.6), where $\rho_{i'}$ has to be replaced by the induced homomorphism $\rho_{i'\,*}$. Treating the 2nd term in (3.6) in a similar way, we obtain the asserted formula. □

Next, we construct cell decompositions of the double quotients (3.1) which meet the assumptions of Theorem 3.2. We shall start with $T\backslash\mathrm{U}(3)/T$.

4 The double quotient $T\backslash\mathrm{U}(3)/T$

Perhaps the most obvious way to treat the double quotient $T\backslash\mathrm{U}(3)/T$ is to view it as the quotient of the left T-action on the flag manifold $\mathrm{U}(3)/T$ and to construct a cell decomposition of this quotient from the Schubert cells of $\mathrm{U}(3)/T$ associated with Borel subgroups of $\mathrm{GL}(3,\mathbb{C})$ that contain T. However, we will not follow this road. Instead of working with Schubert cells, we will relate the double quotient $T\backslash\mathrm{U}(3)/T$ with the bistochastic and unistochastic (3×3)-matrices and define the cells directly by conditions on the entries of the matrices they contain. The relation between the cells so constructed and the Schubert cells of $\mathrm{U}(3)/T$ will be clarified in Appendix B.

We start with introducing some notation. In this section we use the shorthand notation $Y := T\backslash\mathrm{U}(3)/T$. Let $Q \subseteq Y$ denote the subset of classes that have real representatives, i.e., which intersect $\mathrm{O}(3)$. Let $\mathcal{B}_3$ denote the set of (3×3)-matrices with real nonnegative entries that add up to 1 in each row and each column. Such a matrix is called bistochastic. $\mathcal{B}_3$ has the structure of a convex polytope. It is known as the Birkhoff polytope of rank 3. The corners of this polytope are given by the permutation matrices. There exists a natural map $\psi : \mathrm{U}(3) \to \mathcal{B}_3$, given by

$$\psi(a)_{ij} := |a_{ij}|^2\,. \quad (4.1)$$

A point in the image of ψ is called a unistochastic matrix. A point in the image of the restriction of ψ to $\mathrm{O}(3)$ is called an orthostochastic matrix. The subsets of unistochastic and orthostochastic matrices are denoted by $\mathcal{U}_3$ and $\mathcal{O}_3$, respectively. Bistochastic and unistochastic matrices have several applications in mathematics, computer science and physics, see the introduction of [3] for a brief overview. It is known that $\mathcal{U}_3$ is a closed star-shaped 4-dimensional subset of $\mathcal{B}_3$ and that $\mathcal{O}_3$ is its boundary [3, Thm. 3]. Hence, topologically, $\mathcal{U}_3$ is a 4-disk and $\mathcal{O}_3$ is a 3-sphere. To make this information available for the study of Y, we observe that the map ψ descends to a continuous and surjective map (same notation)

$$\psi : Y \to \mathcal{U}_3 .$$

This map will now be analyzed. For a (3×3)-matrix a, let $\overline{a}$ denote the complex conjugate matrix, i.e., $(\overline{a})_{ij} = \overline{a_{ij}}$. Since $\overline{T} = T$, complex conjugation induces a well-defined map $Y \to Y$ which will be denoted $y \mapsto \overline{y}$, too. This map is a homeomorphism.

Lemma 4.1. *Let $y \in Y$. Then $\overline{y} = y$ if and only if $y \in Q$.*

Proof. We have to show that $\overline{y} = y$ implies $y \in Q$. Let $a \in \mathrm{U}(3)$ be a representative of y. By assumption, there exist $b_1, b_2 \in T$ such that $\overline{a} = b_1 a b_2$. For each b_i there exists $c_i \in T$ such that $c_i^2 = b_i$. Then $\overline{c_1 a c_2} = c_1 a c_2$, i.e., $c_1 a c_2 \in \mathrm{O}(3)$ and hence $y \in Q$. □

Lemma 4.2. *Let $y_1, y_2 \in Y$. If $\psi(y_1) = \psi(y_2)$ then $y_2 = y_1$ or $y_2 = \overline{y_1}$.* □

Proof. Let $y \in Y$. Obviously, $\psi(\overline{y}) = \psi(y)$. Hence, we have to show that y and $\overline{y}$ are the only elements of Y that are mapped to $\psi(y)$ under ψ. The question to what extent a unitary matrix of rank 3 is determined by the moduli of its entries was discussed in [14] in connection with CP-violation in the standard electroweak model. The proof uses the unitarity triangles introduced there. Let a be a representative of y. Up to the action of $T \times T$ we may assume that the entries of the first row and the first column of a are real and nonnegative. Define complex numbers

$$u_i = \overline{a_{1i}} a_{2i} , \qquad u^i = \overline{a_{i1}} a_{i2} , \qquad v_i = \overline{a_{1i}} a_{3i} , \qquad v^i = \overline{a_{i1}} a_{i3} , \qquad i,j = 1,2,3 .$$

Unitarity implies $\sum_i u_i = \sum_i u^i = \sum_i v_i = \sum_i v^i = 0$. Hence, the triples (u_1, u_2, u_3), (u^1, u^2, u^3), (v_1, v_2, v_3), (v^1, v^2, v^3) form triangles in the complex plane, with one side on the real axis. These triangles are called unitarity triangles. They are possibly degenerated to a line. The following data of these triangles are determined by y and the choice of a: the side on the real axis, the length of the other two sides, the order of sides. The crucial observation is that these data fix the triangles up to complex conjugation, i.e., up to the transformations $(u_1, u_2, u_3) \mapsto (\overline{u_1}, \overline{u_2}, \overline{u_3})$ etc. . Now, taking the complex conjugate of (u_1, u_2, u_3) requires taking the complex conjugate of the entries a_{22} and a_{23}, hence implies taking the complex conjugate of (u^1, u^2, u^3). Iterating this argument we find that one has to take the complex conjugate of all the triangles, and hence of all the entries of a, simultaneously. This proves the lemma. □

Lemma 4.3. *The map ψ is open.*

Proof. Since ψ maps from a Hausdorff space to a compact space, it is closed. Since it is also surjective, it maps open subsets that are saturated, i.e., consist of full pre-images, to open subsets. Hence, it suffices to show that the saturation of an open subset M is open. By Lemma 4.2, the saturation is given by $M \cup \overline{M}$. Since the map $y \mapsto \overline{y}$ is a homeomorphism, $M \cup \overline{M}$ is open. $\square$

Proposition 4.4. *By restriction, ψ induces a 2-fold covering $Y \setminus Q \to \mathcal{U}_3 \setminus \mathcal{O}_3$ and a homeomorphism $Q \to \mathcal{O}_3$.*

Remark 1. This 2-fold covering carries in fact the structure of a locally trivial principal fibre bundle with structure group $\mathbb{Z}_2$, acting by conjugation.

Proof. By construction, the maps are well-defined and surjective. According to Lemmas 4.1 and 4.2, ψ is injective on Q. By Lemma 4.3, it is then a homeomorphism. To check that the restriction of ψ to $Y \setminus Q$ yields a 2-fold covering, let $y \in Y \setminus Q$ and denote $u = \psi(y)$. By Lemma 4.1, $y \neq \overline{y}$. Since Y is Hausdorff, there exist disjoint open neighbourhoods V_1 of y and V_2 of $\overline{y}$. Let $V := V_1 \cap \overline{V}_2$ and $U := \psi(V)$. By construction, V and $\overline{V}$ are disjoint open neighbourhoods of y and $\overline{y}$, respectively. By Lemma 4.2, $\psi^{-1}(U) = V \cup \overline{V}$. Lemma 4.3 implies that U is an open neighbourhood of u and that, by restriction, ψ induces homeomorphisms V and $\overline{V}$ onto U. $\square$

Since $\mathcal{U}_3$ is a 4-disk with boundary $\mathcal{O}_3$, $Y \setminus Q$ consists of 2 connected components, each of which is a copy of the open 4-disk. Denote these connected components by $Y_\pm$. Since Q is closed, the $Y_\pm$ are open in Y. According to Proposition 4.4, by restriction, ψ induces a homeomorphism of $Y_\pm \cup Q$ onto $\mathcal{U}_4$. In particular, $Y_\pm$ is dense in $Y_\pm \cup Q$. As $Y_\pm \cup Q$ has complement $Y_\mp$ in Y, it is closed. Hence, the closure $\overline{Y_\pm}$ of $Y_\pm$ in Y is given by

$$\overline{Y_\pm} = Y_\pm \cup Q .$$

Thus, $\overline{Y_\pm}$ is a 4-disk whose boundary is given by Q. The main conclusion we draw from this is that any cell decomposition of Q combines with the two 4-cells $\overline{Y_\pm}$ to a cell decomposition of Y. We denote $K^4_\pm := \overline{Y_\pm}$.

Remark 2. In addition, it follows that $T\backslash \mathrm{U}(3)/T$ is homeomorphic to a 4-sphere. This information does not help however in the construction of a cell decomposition that meets the requirement that the factorization maps are cellular.

We will now construct a cell decomposition of $Q \cong \mathcal{O}_3$. The construction is based on the observation that under the factorization maps classes of permutation matrices in $T\backslash \mathrm{U}(3)/T$ get identified with one another in a variety of patterns. Therefore, we take the permutation matrices as the 0-cells. Denote them by $K^0_\tau := \{\tau\}$, $\tau \in \mathcal{S}_3$.

1-*skeleton:* Define

$$K^1_{ij} := \{b \in \mathcal{B}_3 : b_{ij} = 1\} , \qquad i,j = 1,2,3 .$$

Parameterisations of these subsets are given by

$$b_{i_+ j_+} = b_{i_- j_-} = t , \qquad b_{i_+ j_-} = b_{i_- j_+} = 1 - t , \qquad t \in [0,1] .$$

Explicitly, for $i = j = 1$,

$$b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 1-t \\ 0 & 1-t & t \end{bmatrix}, \qquad t \in [0,1] .$$

We read off:
1. $K^1_{ij} \subseteq \mathcal{O}_3$: A representing orthogonal matrix a for b is given by $a_{ij} = \pm\sqrt{b_{ij}}$ with appropriately chosen signs. E.g., for the case $i = j = 1$,

$$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{t} & -\sqrt{1-t} \\ 0 & \sqrt{1-t} & \sqrt{t} \end{bmatrix}, \qquad t \in [0,1] .$$

2. For given i, j, there are two permutations mapping j to i. An explicit calculation yields that these permutations are given by $\tau_i \tau_j \tau_i$ and $\tau_j \tau_i$. The first one is odd, the second one is even. K^1_{ij} is the line in the vector space of real (3×3)-matrices connecting the permutation matrices that correspond to these two permutations. Hence, the K^1_{ij} are 1-disks and

$$\partial K^1_{ij} = K^0_{\tau_i \tau_j \tau_i} \cup K^0_{\tau_j \tau_i} . \tag{4.2}$$

3. $K^1_{ij} \cap K^1_{i_\pm j_\pm} = K^0_{\tau_+}$ where τ_+ is the even permutation mapping j to i and $K^1_{ij} \cap K^1_{i_\pm j_\mp} = K^0_{\tau_-}$ where τ_- is the odd permutation mapping j to i. All the other intersections are trivial. Thus, the cells K^0_τ and K^1_{ij} so constructed, together with the obvious attaching maps, form a cell complex of dimension 1.

2-*skeleton:* For $i, j = 1, 2, 3$, define

$$K^2_{ij} := \{ b \in \mathcal{U}_3 : b_{ij} = 0 \} .$$

To be definite, the following argument is given for K^2_{11}. It easily carries over to the other K^2_{ij}.

Lemma 4.5. *The map*

$$[0,1]^2 \to \mathcal{B}_3 , \quad (s,t) \mapsto \begin{bmatrix} 0 & t & 1-t \\ s & (1-s)(1-t) & (1-s)t \\ 1-s & s(1-t) & st \end{bmatrix} \tag{4.3}$$

induces a homeomorphism of $[0,1]^2$ *onto* K^2_{11}.

Proof. The map is injective and closed. Hence, it suffices to check that its image coincides with K^2_{11}. Denote the image by $\mathcal{I}$. According to [2, 14], a bistochastic (3×3)-matrix is unistochastic iff for two arbitrarily chosen rows or columns the 'chain-links' condition is satisfied. For the 1st and 2nd column this condition reads

$$\left| \sqrt{b_{21}}\sqrt{b_{22}} - \sqrt{b_{31}}\sqrt{b_{32}} \right| \leq \sqrt{b_{11}}\sqrt{b_{12}} \leq \sqrt{b_{21}}\sqrt{b_{22}} + \sqrt{b_{31}}\sqrt{b_{32}} .$$

In case $b_{11} = 0$ this yields

$$b_{21}b_{22} = b_{31}b_{32} \,. \tag{4.4}$$

Since this condition is satisfied for the elements of $\mathcal{I}$, $\mathcal{I} \subseteq \mathcal{U}_3$ and hence $\mathcal{I} \subseteq K^2_{11}$. Conversely, let $b \in K^2_{11}$. Then

$$b = \begin{bmatrix} 0 & t & 1-t \\ s & b_{22} & b_{23} \\ 1-s & b_{32} & b_{33} \end{bmatrix}$$

for some $s, t \in [0,1]$. Since $K^2_{11} \subseteq \mathcal{U}_3$, (4.4) holds. Hence, $sb_{22} = (1-s)b_{32}$. Together with $t + b_{22} + b_{32} = 1$, this yields $b_{22} = (1-s)(1-t)$ and $b_{32} = s(1-t)$. Then $b_{32} = (1-s)t$ and $b_{33} = st$. Hence, $K^2_{11} \subseteq \mathcal{I}$. $\square$

We deduce:
1. $K^2_{ij} \subseteq \mathcal{O}_3$. A representing orthogonal matrix a is given by $a_{ij} = \pm\sqrt{b_{ij}}$ with appropriately chosen signs. E.g., for $i, j = 1$, in the parameterisation (4.3),

$$a = \begin{bmatrix} 0 & \sqrt{t} & \sqrt{1-t} \\ \sqrt{s} & \sqrt{(1-s)(1-t)} & -\sqrt{(1-s)t} \\ \sqrt{1-s} & -\sqrt{s(1-t)} & -\sqrt{st} \end{bmatrix} .$$

2. The K^2_{ij} are 2-disks. For the case of K^2_{11}, the boundary is obtained by setting $s = 0, 1$ or $t = 0, 1$ in (4.3). For the general case this yields

$$\partial K^2_{ij} = K^1_{i\,j_+} \cup K^1_{i\,j_-} \cup K^1_{i_+\,j} \cup K^1_{i_-\,j} \,. \tag{4.5}$$

3. The intersection of K^2_{11} with any other K^2_{ij} consists of elements of K^2_{11} with two zero entries. From the parameterisation (4.3) we see that these elements must have $s = 0, 1$ or $t = 0, 1$. Hence, the intersection is a union of 1-cells. It is obvious that this holds for any intersection of two distinct K^2_{ij}.
Thus, the 2-cells K^2_{ij}, together with the 1-skeleton and the obvious attaching maps, yield a cell complex of dimension 2.

3-*skeleton:* Let $\tau \in \mathcal{S}_3$. Consider the subcomplex of the 2-skeleton consisting of cells that do not intersect K^0_τ. According to (4.2) and (4.5) these cells are

$$K^0_\sigma \,, \quad \sigma \neq \tau \,, \qquad K^1_{ij} \,, \quad i \neq \tau(j) \,, \qquad K^2_{ij} \,, \quad i = \tau(j) \,. \tag{4.6}$$

As $\mathcal{O}_3$ is a 3-sphere, the subset $\mathcal{O}_3 \setminus K^0_\tau$ can be mapped homeomorphically onto $\mathbb{R}^3$. This way, the subcomplex (4.6) is embedded into $\mathbb{R}^3$. A simple inspection of the boundaries of the 1 and 2-cells then shows that this subcomplex is homeomorphic to a 2-sphere. As an illustration, for the case of $\tau = \tau_1$ and for an appropriately chosen homeomorphism $\mathcal{O}_3 \setminus K^0_{\tau_1} \cong \mathbb{R}^3$, the 1-skeleton of this subcomplex is shown in Figure 1.
It follows that the subcomplex (4.6) cuts out a subset of $\mathcal{O}_3$ homeomorphic to the 3-disk. Denote this subset by K^3_τ. We take K^3_τ, $\tau \in \mathcal{S}_3$, as the 3-cells. By construction,

$$\partial K^3_\tau = K^2_{\tau(1)\,1} \cup K^2_{\tau(2)\,2} \cup K^2_{\tau(3)\,3} \,, \tag{4.7}$$

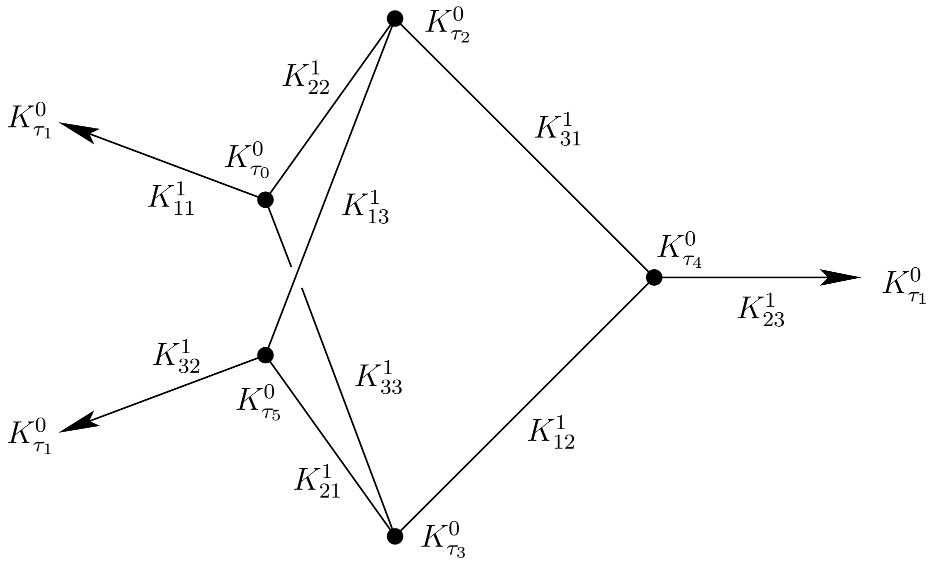

Figure 1: The subcomplex of the 2-skeleton consisting of cells that do not intersect $K^0_{\tau_1}$. The 2-cells form the faces of the triangular double pyramid.

and the intersection of two distinct 3-cells is a union of 2-cells. Thus, the K^3_τ, together with the 2-skeleton, form a cell complex of dimension 3. Since $\mathcal{O}_3$ coincides with the union of the 3-cells, we thus have constructed a cell decomposition of $\mathcal{O}_3$. As stated above, together with $K^4_\pm$, this yields a cell decomposition of Y.

Remark 3. The construction of cells is inspired by the polytope structure of $\mathcal{B}_3$. The relation between the cells of $\mathcal{O}_3$ and the faces of $\mathcal{B}_3$ is as follows. The 0-cells of $\mathcal{O}_3$ coincide with the corners of $\mathcal{B}_3$. The 1-cells of $\mathcal{O}_3$ are those edges of $\mathcal{B}_3$ that are contained in $\mathcal{O}_3$. Since in $\mathcal{B}_3$ there is an edge between any two corners, there are 6 more edges in $\mathcal{B}_3$ that are not orthostochastic. Since any 2-face of $\mathcal{B}_3$ contains one of these non-orthostochastic edges, the 2-faces of $\mathcal{B}_3$ do not have nontrivial intersection with $\mathcal{O}_3$. Instead, the 2-cells of $\mathcal{O}_3$ coincide with the intersections of the 3-faces ('facets') of $\mathcal{B}_3$ with $\mathcal{O}_3$. The 3-cells of $\mathcal{O}_3$ do not have an analogue in $\mathcal{B}_3$.

Next, we have to choose an orientation of the cells and to compute the boundary operator.

Proposition 4.6. *There exists an orientation of cells of Y such that the boundary operator is given by*

$$\partial K^1_{ij} = K^0_{\tau_i\tau_j\tau_i} - K^0_{\tau_j\tau_i}\,, \qquad \partial K^2_{ij} = K^1_{ij+} + K^1_{ij-} - K^1_{i+j} - K^1_{i-j} = \sum_{l=1}^3 K^1_{il} - K^1_{lj}\,,$$
$$\partial K^3_\tau = \operatorname{sign}(\tau)\ \sum_{i=1}^3\ K^2_{\tau(i)\,i}\,, \qquad \partial K^4_\pm = \pm\sum_{\tau\in\mathcal{S}_3}\ K^3_\tau\,.$$

Proof. We define the orientations inductively as follows. For a cell in dimension $n+1$ we choose a certain boundary n-cell and require the boundary orientation of this n-cell to coincide with its genuine orientation chosen before.

1-*cells:* Since according to (4.2), each 1-cell connects 0-cells of opposite sign, we can choose the 0-cells labelled by an odd permutation as starting points. This yields the asserted formula for the 1-cells.

2-*cells:* Due to (4.5), K^2_{ij} is bounded by four 1-cells. Since neighbouring boundary 1-cells intersect in either their starting point or their end-point, they would induce opposite orientations on K^2_{ij}. Hence, opposite boundary 1-cells induce the same orientation. Since there does not exist a permutation mapping both j_+ and j_- to i, $K^1_{i\,j_+}$ and $K^1_{i\,j_-}$ are opposite. We choose the orientation of K^2_{ij} to be induced from these boundary 1-cells (i.e., those with coinciding first index). Since then $K^1_{i_+\,j}$ and $K^1_{i_-\,j}$ are opposite, too, the formula for ∂K^2_{ij} follows.

3-*cells:* The boundary 2-cells of K^3_τ are given by (4.7). Consider a 1-cell K^1_{kl} that belongs to two of these boundary 2-cells, say $K^2_{\tau(i)\,i}$ and $K^2_{\tau(j)\,j}$, where $i \neq j$. Then $k = \tau(i)$ or $l = i$ and $k = \tau(j)$ or $l = j$. Since $i \neq j$, then either $k = \tau(i)$ and $l = j$ or $k = \tau(j)$ and $l = i$. In both cases, the boundary orientations induced on K^1_{kl} from $K^2_{\tau(i)\,i}$ and $K^2_{\tau(j)\,j}$ are opposite. It follows that all three boundary 2-cells would induce the same orientation on K^3_τ. We could simply choose the orientations induced this way for all the 3-cells. However, these orientations would obviously not combine to an orientation of the 3-sphere Q, because the orientations of intersecting 3-cells would be opposite. Instead, we choose the orientations as follows. Consider a 2-cell K^2_{ij}. By (4.7), K^2_{ij} belongs to all 3-cells labelled by a permutation that maps j to i. As noted above, these permutations are given by $\tau_j\tau_i$ and $\tau_i\tau_j\tau_i$. Hence, each 2-cell belongs to one 3-cell labelled by an even permutation and to one 3-cell labelled by an odd permutation. Thus, if we choose the orientation of K^3_τ to be induced from its boundary 2-cells when τ is even and to be opposite to that when τ is odd, these orientations combine to an orientation of Q. The formula for the boundary operator then follows.

4-*cells:* By construction, all boundary 3-cells of $K^4_\pm$ would induce the same orientation on $K^4_\pm$. In order that the orientations of K^4_+ and K^4_- combine to an orientation of Y, we choose that of K^4_+ to be induced from its boundary 3-cells and that of K^4_- to be opposite. This yields the asserted formula. □

Remark 4. The 2-skeleton can be conveniently visualized in a diagram, see Figure 2. In this diagram, six of the nine 2-cells have the shape of a trapezium and three of them form a rectangle with one pair of opposite edges being twisted once. They are labelled by symbols which mimic their shape.

5 The other double quotients

We now turn to the description of the other double quotients and the associated factorization maps. We use the following notation. The standard 2-simplex is denoted by σ^2. If we label its vertices by $1, 2, 3$, the edges are given, in the standard notation, by [12], [23] and [31] and the vertices are given by [1], [2], [3]. The edge $[ij]$ is oriented from $[i]$ to $[j]$.

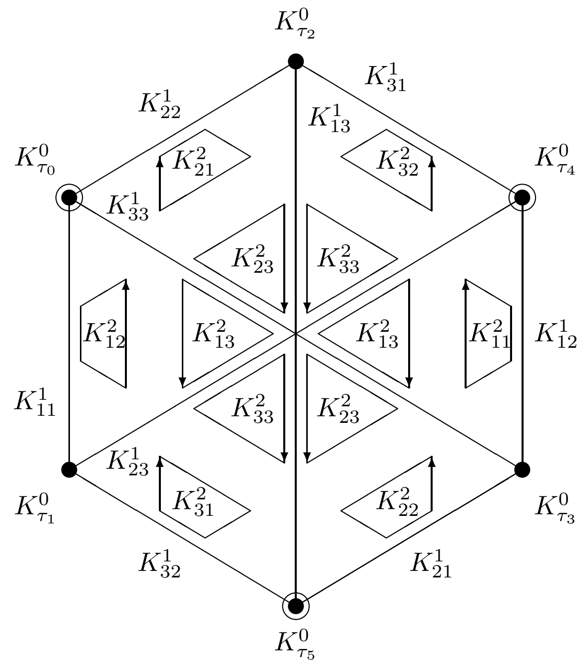

Figure 2: The 2-skeleton of Y with orientation. The starting points of the 1-cells are encircled. The orientations of the 2-cells are indicated by an arrow along the boundary of the corresponding symbols. Note that the crossing point at the center is not a vertex.

Similarly, the standard 1-simplex is denoted by σ^1 and its vertices by [1] and [2]. σ^1 is oriented from [1] to [2]. There is an obvious ambiguity in this notation. However, whether [1] and [2] denote vertices of σ^1 or σ^2 will always be clear from the context.
Define maps $\psi_i : \mathcal{U}_3 \to \sigma^2$, $\psi^i : \mathcal{U}_3 \to \sigma^2$ and $\psi_{ij} : \mathcal{U}_3 \to \sigma^1$ by

$$\psi_i(b) = (b_{i1}, b_{i2}, b_{i3})\,, \qquad \psi^i(b) = (b_{1i}, b_{2i}, b_{3i})\,, \qquad \psi_{ij}(b) = (b_{ij}, 1 - b_{ij})\,.$$

Lemma 5.1. *The maps $\psi_i \circ \psi$, $\psi^i \circ \psi$ and $\psi_{ij} \circ \psi$ descend to homeomorphisms $U_i \backslash \mathrm{U}(3)/T \to \sigma^2$, $T\backslash \mathrm{U}(3)/U_i \to \sigma^2$ and $U_i\backslash \mathrm{U}(3)/U_j \to \sigma^1$, respectively. If we use these homeomorphisms to identify the quotients with the corresponding simplices, the factorization maps* (3.2) *satisfy*

$$\lambda_i^{12} = \psi_i \circ \psi\,, \qquad \lambda_i^{21} = \psi^i \circ \psi\,, \qquad \mu_{ij}^{12} \circ \lambda_i^{12} = \mu_{ij}^{21} \circ \lambda_j^{21} = \psi_{ij} \circ \psi\,. \tag{5.1}$$

Proof. First we have to check that the maps descend to the quotients. For $\psi_i \circ \psi$, this follows from the observation that the action of U_i and T on $\mathrm{U}(3)$ by left multiplication do not differ in their effect on the i-th row of a matrix. Similarly, for $\psi^i \circ \psi$, the action of U_i and T on $\mathrm{U}(3)$ by right multiplication do not differ in their effect on the i-th column. Combining these two observations we obtain the assertion for $\psi_{ij} \circ \psi$.
Next, we show that the descended maps are 1:1. As maps from a Hausdorff space to a compact space they are homeomorphisms then. Surjectivity is obvious. To check injectivity

of the map $U_i\backslash\mathrm{U}(3)/T \to \sigma^2$ induced by $\psi_i \circ \psi$, let $a, b \in \mathrm{U}(3)$ such that $\psi_i \circ \psi(TaT) = \psi_i \circ \psi(TbT)$. Then $b_{ij} = \alpha_j a_{ij}$ for some $\alpha_j \in \mathrm{U}(1)$, $j = 1, 2, 3$. Hence, up to the action of T by right multiplication, we may assume that the i-th rows of a and b coincide. Consider $c := ba^{-1}$. We have $b = ca$ and $c_{ii} = 1$, because this is the scalar product of the i-th row of b with the i-th row of a. Hence, $c \in U_i$ and a and b define the same class in the quotient $U_i\backslash\mathrm{U}(3)/T$. For the map $T\backslash\mathrm{U}(3)/U_i \to \sigma^2$ induced by $\psi^i \circ \psi$, the argument is analogous. For the map $U_i\backslash\mathrm{U}(3)/U_j \to \sigma^1$ induced by $\psi_{ij} \circ \psi$, let $a, b \in \mathrm{U}(3)$ such that $\psi_{ij} \circ \psi(TaT) = \psi_{ij} \circ \psi(TbT)$. Then $b_{ij} = \alpha a_{ij}$ for some $\alpha \in \mathrm{U}(1)$. By the action of U_j by right multiplication, the entry a_{ij} can be multiplied by an arbitrary phase and the other two entries in the i-th row of a can be arbitrarily adjusted. Hence, up to this action, we may assume that the i-th rows of a and b coincide. Then the same argument as for $\psi_i \circ \psi$ yields that a and b belong to the same class in the quotient $U_i\backslash\mathrm{U}(3)/U_j$.
Finally, the equalities (5.1) hold by construction. □

Remark 5. A slightly different interpretation of the quotients $T\backslash\mathrm{U}(3)/U_i$ and $U_i\backslash\mathrm{U}(3)/T$ is obtained as follows. Extraction of the i-th row defines a map from $\mathrm{U}(3)$ to the 5-sphere S^5 that translates the action of T on $\mathrm{U}(3)$ by left multiplication into the natural action of $T \cong \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$ on $\mathrm{S}^5 \subseteq \mathbb{C}^3$. This map descends to a homeomorphisms of $\mathrm{U}(3)/U_i$ onto complex projective space $\mathbb{C}\mathrm{P}^2$ that translates the action of T on $\mathrm{U}(3)/U_i$ into the action of T on $\mathbb{C}\mathrm{P}^2$ inherited from the natural action of T on $\mathbb{C}^3$. Thus, the quotient $T\backslash\mathrm{U}(3)/U_i$ may be identified with the quotient of $\mathbb{C}\mathrm{P}^2$ w.r.t. this action. A similar result holds for the quotient $U_i\backslash\mathrm{U}(3)/T$.

Now we use (5.1) to compute how the factorization maps λ_i^{12}, λ_i^{21}, μ_{ij}^{12} and μ_{ij}^{12} map the cells of $T\backslash\mathrm{U}(3)/T$ and σ^2, respectively.

Proposition 5.2. *The factorization maps* (3.2) *map the cells of* $T\backslash\mathrm{U}(3)/T$ *and* σ^2 *as follows:*

	K^0_τ	K^1_{jk}	K^2_{jk}	$K^3_\tau, K^4_\pm$
λ_i^{12}	$[\tau^{-1}(i)]$	$[k] \mid j = i$ $[k_+\, k_-] \mid j \neq i$	$[k_+\, k_-] \mid j = i$ $\sigma^2 \mid j \neq i$	σ^2
λ_i^{21}	$[\tau(i)]$	$[j] \mid k = i$ $[j_+\, j_-] \mid k \neq i$	$[j_+\, j_-] \mid k = i$ $\sigma^2 \mid k \neq i$	σ^2

	$[k]$	$[k\, k_+]$
μ_{ij}^{12}	$[1] \mid k = j$ $[2] \mid k \neq j$	$[2] \mid k = j_+$ $\sigma^1 \mid k \neq j_+$
μ_{ij}^{21}	$[1] \mid k = i$ $[2] \mid k \neq i$	$[2] \mid k = i_+$ $\sigma^1 \mid k \neq i_+$

Here $i, j, k = 1, 2, 3$ and $\tau \in \mathcal{S}_3$. In particular, these maps are cellular. Their induced homomorphisms are given by

$$
\begin{aligned}
&{\lambda_i^{12}}_*(K^0_\tau) = [\tau^{-1}(i)]\,, && {\lambda_i^{12}}_*(K^1_{jk}) = (\delta_{i-,j} - \delta_{i+,j})[k_+\, k_-]\,, && {\lambda_i^{12}}_*(K^2_{jk}) = (\delta_{i-,j} - \delta_{i+,j})\sigma^2\,,\\
&{\lambda_i^{21}}_*(K^0_\tau) = [\tau(i)]\,, && {\lambda_i^{21}}_*(K^1_{jk}) = (\delta_{i-,k} - \delta_{i+,k})[j_+\, j_-]\,, && {\lambda_i^{21}}_*(K^2_{jk}) = (\delta_{i+,k} - \delta_{i-,k})\sigma^2\,,
\end{aligned}
$$

$$
\begin{aligned}
&{\mu_{ij}^{12}}_*([k]) = \delta_{jk}[1] + (1-\delta_{jk})[2]\,, && {\mu_{ij}^{12}}_*([k\, k_+]) = (\delta_{k\,j} - \delta_{k\,j-})\sigma^1\,,\\
&{\mu_{ij}^{21}}_*([k]) = \delta_{ik}[1] + (1-\delta_{ik})[2]\,, && {\mu_{ij}^{21}}_*([k\, k_+]) = (\delta_{k\,i} - \delta_{k\,i-})\sigma^1\,.
\end{aligned}
$$

Cells not listed here are annihilated.

Proof. First, consider λ_i^{12}. For the 0 and 1-cells, the formulae can be read off from the definition of the cells and the definition of ψ_i. Due to $\psi(K^4_\pm) = \mathcal{U}_3$, we have $\lambda_i^{12}(K^4_\pm) = \sigma^2$. To check the formula for the 2-cells, consider $\lambda_i^{12}(K^2_{jk}) = \psi_i \circ \psi(K^2_{jk})$. From the parameterisation (4.3) of K^2_{11}, carried over to K^2_{jk}, we read off: if $i = j$ then the image coincides with the edge of σ^2 where the k-th entry is zero, i.e., with $[k_+\, k_-]$. If $i \neq j$, the image is the whole of σ^2. This yields the formula for the 2-cells. The formula for the 3-cells then follows by observing that each of them contains a 2-cell with first index being different from i. For λ_i^{12}, the argument is similar.
To find μ_{ij}^{12}, for each cell of σ^2 we choose a preimage under λ_i^{12} and determine its image under $\mu_{ij}^{12} \circ \lambda_{ij}^{12} = \psi_{ij} \circ \psi$. For the vertex $[k]$, a preimage under λ_i^{12} is given by K^0_τ where τ obeys $\tau(k) = i$. The entries of the corresponding permutation matrix are $\tau_{ij} = 1$ if $i = \tau(j)$, hence if $j = k$, and 0 otherwise. Hence, $\psi_{ij} \circ \psi(K^0_\tau) = [1]$ if $j = k$ and $[2]$ otherwise. For the edge $[k\, k_+]$, a preimage under λ_i^{12} is given by $K^2_{i\,k_-}$. By $\psi_{ij} \circ \psi$, this subset is mapped to $[2]$ if $j = k_-$ and to σ^1 otherwise. Again, for μ_{ij}^{21}, the argument is similar.
Finally, consider the induced homomorphisms. They annihilate cells that are mapped to lower-dimensional cells by the original factorization maps. To determine the signs, we have to compare the orientations of the cells and their images. For the 1-cells this can be done by finding out whether the starting point of the cell is mapped to the starting point of the image or not. 2-cells are only relevant for $T\backslash \mathrm{U}(3)/T$. For a given 2-cell we choose a boundary 1-cell from which the orientation is induced. Since the 1-cells of σ^2 carry the induced boundary orientation, the 2-cell acquires the same sign as this boundary 1-cell. □

6 Cell complex structure

We now combine the pairs of cells of $\mathcal{A}$ with the cell decompositions of the corresponding double quotients as prescribed by Theorem 3.2. The list of cells, together with their shorthand notation C^{pqr}_{ijk} introduced in the proof of this theorem, is given in the following table. The number of cells in dimension $0, \ldots, 8$ is 9, 18, 24, 27, 24, 15, 9, 6, 2, respectively,

their total number is 134.

dim	cells ($\tau \in \mathcal{S}_3\,,\ i,j = 1,2,3\,,\ a = 1,2$)		
8	$C^{224}_{\pm} = \mathcal{A}^2 \times \mathcal{A}^2 \times K^4_{\pm}$		
7	$C^{223}_{\tau} = \mathcal{A}^2 \times \mathcal{A}^2 \times K^3_{\tau}$		
6	$C^{222}_{ij} = \mathcal{A}^2 \times \mathcal{A}^2 \times K^2_{ij}$		
5	$C^{221}_{ij} = \mathcal{A}^2 \times \mathcal{A}^2 \times K^1_{ij}\,,$	$C^{212}_{i} = \mathcal{A}^2 \times \mathcal{A}^1_i \times \sigma^2\,,$	$C^{122}_{i} = \mathcal{A}^1_i \times \mathcal{A}^2 \times \sigma^2$
4	$C^{220}_{\tau} = \mathcal{A}^2 \times \mathcal{A}^2 \times K^0_{\tau}\,,$	$C^{211}_{ij} = \mathcal{A}^2 \times \mathcal{A}^1_i \times [j_+\, j_-]\,,$	$C^{121}_{ij} = \mathcal{A}^1_i \times \mathcal{A}^2 \times [j_+\, j_-]$
3	$C^{210}_{ij} = \mathcal{A}^2 \times \mathcal{A}^1_i \times [j]\,,$	$C^{120}_{ij} = \mathcal{A}^1_i \times \mathcal{A}^2 \times [j]\,,$	$C^{111}_{ij} = \mathcal{A}^1_i \times \mathcal{A}^1_j \times \sigma^1$
2	$C^{200}_{i} = \mathcal{A}^2 \times \mathcal{A}^0_i \times *\,,$	$C^{020}_{i} = \mathcal{A}^0_i \times \mathcal{A}^2 \times *\,,$	$C^{110}_{i\,j\,a} = \mathcal{A}^1_i \times \mathcal{A}^1_j \times [a]$
1	$C^{100}_{ij} = \mathcal{A}^1_i \times \mathcal{A}^0_j \times *\,,$	$C^{010}_{ij} = \mathcal{A}^0_i \times \mathcal{A}^1_j \times *$	
0	$C^{000}_{ij} = \mathcal{A}^0_i \times \mathcal{A}^0_j \times *$		

Theorem 6.1. *The cells C^{pqr}_{ijk}, together with the characteristic maps induced by the projections π^{pq}_{ij}, see (3.3), define a cell complex structure on $\mathcal{X}$. The boundary operator is given by*

$$\partial C^{224}_{\pm} = \pm \textstyle\sum_{m=0}^{5} C^{223}_{m}\ , \qquad \partial C^{223}_{\tau} = \mathrm{sign}(\tau)\ \textstyle\sum_{i=1}^{3} C^{222}_{i\,\tau(i)}\,,$$

$$\partial C^{222}_{ij} = C^{221}_{ij_+} + C^{221}_{ij_-} - C^{221}_{i_+ j} - C^{221}_{i_- j} + C^{122}_{i+} - C^{122}_{i-} + C^{212}_{i+} - C^{212}_{i-}\,,$$

$$\partial C^{221}_{ij} = C^{220}_{\tau_i\tau_j\tau_i} - C^{220}_{\tau_j\tau_i} + C^{121}_{i_+ j} - C^{121}_{i_- j} + C^{211}_{j_+ i} - C^{211}_{j_- i}\,,$$

$$\partial C^{212}_{i} = -\textstyle\sum_{j=1}^{3} C^{211}_{ij}\ , \qquad \partial C^{122}_{i} = -\textstyle\sum_{j=1}^{3} C^{121}_{ij}\ , \qquad \partial C^{220}_{\tau} = \textstyle\sum_{i=1}^{3} C^{210}_{i\,\tau(i)} + C^{120}_{i\,\tau^{-1}(i)}\,,$$

$$\partial C^{211}_{ij} = C^{210}_{i\,j_+} - C^{210}_{i\,j_-} + C^{111}_{j_+\,i} - C^{111}_{j_-\,i}\ , \qquad \partial C^{121}_{ij} = C^{120}_{i\,j_+} - C^{120}_{i\,j_-} + C^{111}_{i\,j_-} - C^{111}_{i\,j_+}\,,$$

$$\partial C^{210}_{ij} = C^{110}_{j\,i\,1} + C^{110}_{j_+\,i\,2} + C^{110}_{j_-\,i\,2} + C^{200}_{i-} - C^{200}_{i+}\ , \qquad \partial C^{120}_{ij} = -C^{110}_{i\,j\,1} - C^{110}_{i\,j_+\,2} - C^{110}_{i\,j_-\,2} + C^{200}_{i-} - C^{020}_{i+}\,,$$

$$\partial C^{111}_{ij} = C^{110}_{i\,j\,2} - C^{110}_{i\,j\,1}\ , \qquad \partial C^{110}_{i\,j\,a} = C^{100}_{i\,j_+} - C^{100}_{i\,j_-} - C^{010}_{i_+\,j} + C^{010}_{i_-\,j}\ , \qquad \partial C^{200}_{i} = \textstyle\sum_{j=1}^{3} C^{100}_{ji}\,,$$

$$\partial C^{020}_{i} = \textstyle\sum_{j=1}^{3} C^{010}_{ij}\ , \qquad \partial C^{100}_{ij} = C^{000}_{i_-\,j} - C^{000}_{i_+\,j}\ , \qquad \partial C^{010}_{ij} = C^{000}_{i\,j_-} - C^{000}_{i\,j_+}\,.$$

Proof. The first statement is a reformulation of Theorem 3.2. The formulae for the boundary map are a straightforward consequence of the general formula given in the theorem mentioned and the formulae for the factorization maps given in Proposition 5.2. □

7 Stratification

In this section we show that the closures of the orbit type strata are subcomplexes of $\mathcal{X}$ and determine the cells they consist of. We start with introducing some notation and deriving some formulae needed in the sequel.

Let $\tilde{T} := T \cap \mathrm{SU}(3)$ and $\tilde{U}_i := U_i \cap \mathrm{SU}(3)$. We have $\tilde{T} \cong \mathrm{U}(1) \times \mathrm{U}(1)$ and $\tilde{U}_i \cong \mathrm{U}(2)$. Let T_i, $i = 1, 2, 3$ denote the subgroups consisting of the matrices

$$\mathrm{diag}(\overline{\alpha}^2, \alpha, \alpha)\ , \quad \mathrm{diag}(\alpha, \overline{\alpha}^2, \alpha)\ , \quad \mathrm{diag}(\alpha, \alpha, \overline{\alpha}^2)\ ,$$

respectively, where $\alpha \in \mathrm{U}(1)$. The centralizers of the subgroups T_i and $\tilde{U}_i$ in $\mathrm{SU}(3)$ and $\mathrm{U}(3)$ are

$$\mathrm{C}_{\mathrm{U}(3)}(T_i) = U_i\ , \quad \mathrm{C}_{\mathrm{U}(3)}(U_i) = T_i\ , \quad \mathrm{C}_{\mathrm{SU}(3)}(T_i) = \tilde{U}_i\ , \quad \mathrm{C}_{\mathrm{SU}(3)}(\tilde{U}_i) = T_i$$

and their normalizers are

$$\mathrm{N}_{\mathrm{U}(3)}(T_i) = \mathrm{N}_{\mathrm{U}(3)}(\tilde{U}_i) = U_i\ , \quad \mathrm{N}_{\mathrm{SU}(3)}(T_i) = \mathrm{N}_{\mathrm{SU}(3)}(\tilde{U}_i) = \tilde{U}_i\,.$$

Furthermore, there holds

$$\mathcal{A}^1_i = \mathcal{A} \cap T_i\,.$$

Since multiplication of an arbitrary matrix by a permutation matrix from the left or the right results in the corresponding permutation of rows or columns, respectively,

$$\tau U_i = U_{\tau(i)}\tau\ , \quad \tau\,\tilde{U}_i = \tilde{U}_{\tau(i)}\tau\,, \qquad \tau T_i = T_{\tau(i)}\tau\,. \tag{7.1}$$

For $\tau \in \mathcal{S}_3$ and $i, j = 1, 2, 3$, define

$$V^0_\tau := T\tau\,, \qquad V^1_{ij} := \{a \in \mathrm{U}(3) : |a_{ij}| = 1\}\,, \qquad V^2_{ij} := \{a \in \mathrm{U}(3) : a_{ij} = 0\}\,.$$

The subsets V^0_τ, V^1_{ij} and V^2_{ij} consist of the representatives in $\mathrm{U}(3)$ of the elements of the cells K^0_τ, K^1_{ij} and K^2_{ij} of $T\backslash\mathrm{U}(3)/T$, respectively. We have $V^1_{ii} = U_i$ and

$$\tau V^r_{ij} = V^r_{\tau(i)\,j}\,, \qquad V^r_{ij}\tau = V^r_{i\,\tau^{-1}(j)}\,, \qquad r = 1, 2\,. \tag{7.2}$$

Lemma 7.1. $\quad U_i\,\tau\,U_j = \begin{cases} V^1_{ij} & |\ \tau(j) = i\,, \\ V^2_{ij} & |\ \tau(j) \neq i\,. \end{cases}$

Proof. If $\tau(j) = i$, (7.1) and (7.2) yield

$$U_i\,\tau\,U_j = U_i\tau = V^1_{ii}\tau = V^1_{ij}\,.$$

If $\tau(j) \neq i$, there exists a permutation σ such that $\sigma(1) = i$ and $\sigma(2) = \tau(j)$. Under the assumption that there holds $U_1U_2 = V^2_{12}$, (7.1) and (7.2) imply

$$U_i\,\tau\,U_j = \sigma\,U_1U_2\,\sigma^{-1}\tau = \sigma\,V^2_{12}\,\sigma^{-1}\tau = V^2_{ij}\,.$$

Hence, it suffices to show $U_1U_2 = V^2_{12}$. For any $a \in U_1$, $b \in U_2$ we have $(ab)_{12} = \sum_{m=1}^{3} a_{1m}b_{m2} = 0$, because $a_{1m} \neq 0$ only for $m = 1$, whereas $b_{m2} \neq 0$ only for $m = 2$.

Hence, $U_1\,U_2 \subseteq V_{12}^2$. To prove the converse inclusion, let $a \in V_{12}^2$, i.e., $a_{12} = 0$. Since the 2nd column of a is orthogonal to the other two columns,

$$\overline{a_{21}}a_{22} + \overline{a_{31}}a_{32} = 0\ , \qquad \overline{a_{23}}a_{22} + \overline{a_{33}}a_{32} = 0\ . \tag{7.3}$$

We view this as a system of linear equations in the variables a_{22} and a_{32}. Since a does not have a zero row, a_{22} and a_{32} cannot both vanish, so that this system has a nontrivial solution. Hence, the determinant of the (2×2)-matrix

$$\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

vanishes. Then this matrix is a tensor product, i.e., there exist complex numbers c_1, c_2, d_1, d_2 such that $a_{21} = c_1 d_1$, $a_{23} = c_1 d_2$, $a_{31} = c_2 d_1$ and $a_{33} = c_2 d_2$. We can choose c_1, c_2 in such a way that $|c_1|^2 + |c_2|^2 = 1$. Consider the matrices

$$a_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{22} & c_1 \\ 0 & a_{32} & c_2 \end{pmatrix}, \qquad a_2 = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & 1 & 0 \\ d_1 & 0 & d_2 \end{pmatrix}.$$

We have $a_1 a_2 = a$. We check that a_1 is unitary: the columns are obviously unit vectors. According to (7.3), $\overline{d_1}\,(\overline{c_1}a_{22} + \overline{c_2}a_{32}) = 0$ and $\overline{d_2}\,(\overline{c_1}a_{22} + \overline{c_2}a_{32}) = 0$. Since either $d_1 \neq 0$ or $d_2 \neq 0$, it follows that the 2nd and 3rd row of a_1 are orthogonal, so that a_1 is unitary, indeed. Then so is a_2. Thus, $a_1 \in U_1$, $a_2 \in U_2$, and $a \in U_1 U_2$, as asserted. □

Now we turn to the discussion of the orbit type strata of $\mathcal{X}$. The stabilizers of the action of SU(3) on SU(3) × SU(3) by diagonal conjugation are centralizers of pairs in SU(3). It is well known that any subgroup of SU(3) which is a centralizer is conjugate to one of the subgroups $\mathbb{Z}_3$, T_1, $\tilde{T}$, $\tilde{U}_1$, SU(3). The corresponding orbit types will be labelled by the numbers $1, \ldots, 5$ (in the respective order). This numbering reflects the natural ordering of orbit types that is inherited from the natural partial ordering of conjugacy classes of subgroups of SU(3). I.e., type $n \geq$ type n' iff $n \geq n'$. The subset of $\mathcal{X}$ of orbits of type n is denoted by $\mathcal{X}_n$ and its closure in $\mathcal{X}$ by $\overline{\mathcal{X}}_n$. The slice theorem for compact group actions [4] implies that the orbit type subsets $\mathcal{X}_1, \ldots, \mathcal{X}_5$ yield a disjoint decomposition of $\mathcal{X}$ into manifolds satisfying the frontier condition: if $\mathcal{X}_i \cap \overline{\mathcal{X}_j}$ then $\mathcal{X}_i \subseteq \overline{\mathcal{X}_j}$. This orbit type decomposition is in fact a stratification [16]. Therefore we refer to the subsets $\mathcal{X}_i$ as the orbit type strata of $\mathcal{X}$. Due to the ordering of orbit types and the frontier condition,

$$\overline{\mathcal{X}}_5 \subseteq \cdots \subseteq \overline{\mathcal{X}}_1 = \mathcal{X}\,, \qquad \overline{\mathcal{X}}_n = \bigcup\nolimits_{n' \geq n} \mathcal{X}_{n'}\,.$$

In particular, $\mathcal{X}_1$ is the principal stratum.

Remark 6. Subgroups which can be written as a centralizer are called Howe subgroups and pairs of subgroups which centralize each other are known as Howe dual pairs. Such pairs play a prominent role in the representation theory of reductive Lie groups. For an explicit listing of the Howe subgroups of the classical Lie groups together with their partial ordering by inclusion modulo conjugacy, see [19].

Lemma 7.2. *For $(t,s,g) \in \mathcal{A} \times \mathcal{A} \times \mathrm{U}(3)$, the following table lists the conditions on g under which $\varphi(t,s,g)$ belongs to $\overline{\mathcal{X}_n}$, $n = 2, \dots, 5$.*

If (t,s) is in the interior of	$\varphi(t,s,g)$ belongs to $\overline{\mathcal{X}_5}$	$\overline{\mathcal{X}_4}$	$\overline{\mathcal{X}_3}$	$\overline{\mathcal{X}_2}$
$\mathcal{A}^2 \times \mathcal{A}^2$	–	–	$g \in \bigcup_{\tau \in \mathcal{S}_3} V^0_\tau$	$g \in \bigcup_{i,j=1}^3 V^1_{ij}$
$\mathcal{A}^1_i \times \mathcal{A}^2$	–	–	$g \in \bigcup_{j=1}^3 V^1_{ij}$	$g \in \bigcup_{j=1}^3 V^2_{ij}$
$\mathcal{A}^2 \times \mathcal{A}^1_i$	–	–	$g \in \bigcup_{j=1}^3 V^1_{ji}$	$g \in \bigcup_{j=1}^3 V^2_{ji}$
$\mathcal{A}^1_i \times \mathcal{A}^1_j$	–	$g \in V^1_{ij}$	$g \in V^2_{ij}$	all g
$\mathcal{A}^0_i \times \mathcal{A}^2$, $\mathcal{A}^2 \times \mathcal{A}^0_i$	–	–	all g	–
$\mathcal{A}^0_i \times \mathcal{A}^1_j$, $\mathcal{A}^1_i \times \mathcal{A}^0_j$	–	all g	–	–
$\mathcal{A}^0_i \times \mathcal{A}^0_j$	all g	–	–	–

Proof. The orbit type of t, s, g is the conjugacy class of the stabilizer of the pair (t, gsg^{-1}). The stabilizer is

$$S(g) := \mathrm{C}_{\mathrm{SU}(3)}(t) \cap g C_{\mathrm{SU}(3)}(s) g^{-1}\,.$$

$\mathcal{A}^2 \times \mathcal{A}^2$: Let (t,s) be in the interior of $\mathcal{A}^2 \times \mathcal{A}^2$. Then

$$S(g) = \tilde{T} \cap g \tilde{T} g^{-1}\,.$$

In particular, $S(g)$ is a Howe subgroup of SU(3) contained in $\tilde{T}$. Hence, $S(g) = \tilde{T}$ (type 3), T_i (type 2) or $\mathbb{Z}_3$ (type 1).
Case $S(g) = \tilde{T}$: Here $g \in \mathrm{N}_{\mathrm{U}(3)}(\tilde{T}) = \mathrm{N}_{\mathrm{U}(3)}(T) = \bigcup_{\tau \in \mathcal{S}_3} V^0_\tau$.
Case $S(g) = T_i$: Here $g^{-1} T_i g \subseteq \tilde{T}$, i.e., $g^{-1} T_i g$ is a Howe subgroup of SU(3) contained in $\tilde{T}$. Hence, $g^{-1} T_i g = T_j$ for some $j = 1, 2, 3$. Due to (7.1), $T_j = \tau^{-1} T_i \tau$, where $\tau \in \mathcal{S}_3$ is chosen so that $\tau(j) = i$. It follows $g\tau^{-1} \in \mathrm{N}_{\mathrm{U}(3)}(T_i) = U_i$ and hence $g \in \bigcup_{\tau \in \mathcal{S}_3} U_i \cdot \tau$. Conversely, if g is of this form then $g^{-1} T_i g = \tau^{-1} T_i \tau \subseteq \tau^{-1} \tilde{T} \tau = \tilde{T}$, hence $T_i \subseteq S(g)$. Equality holds if g is not in $\bigcup_{\tau \in \mathcal{S}_3} V^0_\tau$. Due to (7.2), $\bigcup_{\tau \in \mathcal{S}_3} U_i \cdot \tau = \bigcup_{\tau \in \mathcal{S}_3} V^1_{ii} \cdot \tau = \bigcup_{j=1}^3 V^1_{ij}$. Since all T_i belong to orbit type 2, we have to take the union over $i = 1, 2, 3$, too.

$\mathcal{A}^1_i \times \mathcal{A}^2$: Let (t,s) be in the interior of $\mathcal{A}^1_i \times \mathcal{A}^2$. Then

$$S(g) = \tilde{U}_i \cap g \tilde{T} g^{-1}\,.$$

This is a Howe subgroup of SU(3) contained in the maximal toral subgroup $g\tilde{T}g^{-1}$. Hence, it can be $g\tilde{T}g^{-1}$ (type 3), gT_jg^{-1}, $j = 1, 2, 3$, (type 2) or $\mathbb{Z}_3$ (type 1).
Case $S(g) = g\tilde{T}g^{-1}$: Under this assumption both $g\tilde{T}g^{-1}$ and $\tilde{T}$ are maximal toral subgroups of $\tilde{U}_i$, hence are conjugate in $\tilde{U}_i$. I.e., there exists $h \in \tilde{U}_i \subseteq U_i$ such that $g\tilde{T}g^{-1} = h\tilde{T}h^{-1}$. Then $h^{-1}g \in \mathrm{N}_{\mathrm{U}(3)}(\tilde{T}) = N_{\mathrm{U}(3)}(T)$, hence $g \in U_i \cdot N_{\mathrm{U}(3)}(\tilde{T}) = \bigcup_{\tau \in \mathcal{S}_3} U_i \cdot \tau = \bigcup_{j=1}^3 V^1_{ij}$. The converse assertion is obvious.

Case $S(g) = gT_jg^{-1}$: Here gT_jg^{-1} is contained in some maximal toral subgroup of $\tilde{U}_i$. Hence, there exists $h \in \tilde{U}_i \subseteq U_i$ such that $h^{-1}gT_jg^{-1}h \subseteq \tilde{T}$. By the same argument as in the case of $\mathcal{A}^2 \times \mathcal{A}^2$ we conclude that $g^{-1}h \in \bigcup_{\tau\in\mathcal{S}_3} U_j \cdot \tau$. Then

$$g \in \bigcup_{\tau\in\mathcal{S}_3} U_i \cdot \tau \cdot U_j \,. \tag{7.4}$$

Conversely, if g is of this form, $gT_jg^{-1} \subseteq \tilde{U}_i$, hence it is contained in $S(g)$, and is equal to $S(g)$ if g is not in $\bigcup_{k=1}^3 V_{ik}^1$. Since all values of j belong to type 2, in (7.4) we have to take the union over $j = 1, 2, 3$. According to Lemma 7.1, this yields $\bigcup_{j=1}^3 \bigcup_{\tau\in\mathcal{S}_3} U_i \cdot \tau \cdot U_j = \bigcup_{j=1}^3 V_{ij}^2$. For (t, s) in the interior of $\mathcal{A}^2 \times \mathcal{A}_i^1$, the proof is analogous.

$\mathcal{A}_i^1 \times \mathcal{A}_j^1$: Let (t, s) be in the interior of $\mathcal{A}_i^1 \times \mathcal{A}_j^1$. Then

$$S(g) = \tilde{U}_i \cap g\tilde{U}_jg^{-1} \,,$$

i.e., $S(g)$ is a Howe subgroup of SU(3) contained in $\tilde{U}_i$. Hence, we can have $S(g) = \tilde{U}_i$, $S(g) \cong \mathrm{U}(1) \times \mathrm{U}(1)$, $S(g) \cong \mathrm{U}(1)$ or $S(g) = \mathbb{Z}_3$.

Case $S(g) = \tilde{U}_i$: Let $\tau \in \mathcal{S}_3$ such that $\tau(j) = i$. According to (7.1), $g\tilde{U}_jg^{-1} = \tilde{U}_i = \tau\tilde{U}_j\tau^{-1}$. It follows $\tau^{-1}g \in \mathrm{N}_{\mathrm{U}(3)}(\tilde{U}_j) = U_j$ and thus $g \in \tau U_j$. The converse implication is obvious. Due to (7.2), $\tau U_j = \tau V_{jj} = V_{ij}$.

Case $S(g) \cong \mathrm{U}(1) \times \mathrm{U}(1)$: Here, $S(g)$ is a maximal toral subgroup in SU(2). There exists $h \in \tilde{U}_i \subseteq U_i$ such that $S(g) = h\tilde{T}h^{-1}$. It follows $\tilde{T} \subseteq h^{-1}g\tilde{U}_jg^{-1}h$. By taking the centralizer in SU(3) we obtain $h^{-1}gT_jg^{-1}h \subseteq \tilde{T}$. As explained for the case of $\mathcal{A}_i^1 \times \mathcal{A}^2$ above, then $g \in \bigcup_{\tau\in\mathcal{S}_3} U_i \cdot \tau \cdot U_j$. Conversely, if g is of this form then $g\tilde{U}_jg^{-1} = h\tilde{U}_kh^{-1}$ for some $k = 1, 2, 3$ and $h \in U_i$. Hence,

$$S(g) = \tilde{U}_i \cap h\tilde{U}_kh^{-1} = h(\tilde{U}_i \cap \tilde{U}_k)h^{-1} \supseteq h\tilde{T}h^{-1} \,,$$

i.e., $S(g)$ contains a subgroup that is isomorphic to $\mathrm{U}(1) \times \mathrm{U}(1)$. Equality holds if g is not in V_{ij}^1. Finally, Lemma 7.1 yields $\bigcup_{\tau\in\mathcal{S}_3} U_i\tau U_j = V_{ij}^2$.

Case $S(g) \cong \mathrm{U}(1)$: For any $t \in \mathcal{A}_i^1$, $s \in \mathcal{A}_j^1$, both t and gsg^{-1} have a degenerate eigenvalue. The intersection of the corresponding eigenspaces contains a nontrivial common eigenvector u. Then the pair (t, gsg^{-1}) is invariant under the U(1)-subgroup of SU(3) defined by multiplying u by α^2 and vectors orthogonal to u by $\overline{\alpha}$. It follows that the case $S(g) = \mathbb{Z}_3$ does not occur, so that for all remaining g the type of $\varphi(t, s, g)$ is 2.

Remaining cases: If $t \in \mathbb{Z}_3$, the orbit type of $\varphi(t, s, g)$ is given by the centralizer of s, and vice versa. □

Theorem 7.3. *The closures of the nonprincipal strata are subcomplexes of $\mathcal{X}$. Their dimensions and the cells they consist of are listed in the following table:*

	dim	cells
$\overline{\mathcal{X}}_5$	0	C_{ij}^{000}, $i, j = 1, 2, 3$
$\overline{\mathcal{X}}_4$	2	cells of $\overline{\mathcal{X}}_5$, C_{ij}^{100}, C_{ij}^{010}, C_{ij1}^{110}, $i, j = 1, 2, 3$
$\overline{\mathcal{X}}_3$	4	cells of $\overline{\mathcal{X}}_4$, C_i^{200}, C_i^{020}, C_{ij2}^{110}, C_{ij}^{210}, C_{ij}^{120}, $i, j = 1, 2, 3$, C_τ^{220}, $\tau \in \mathcal{S}_3$
$\overline{\mathcal{X}}_2$	5	cells of $\overline{\mathcal{X}}_3$, C_{ij}^{111}, C_{ij}^{211}, C_{ij}^{121}, C_{ij}^{221}, $i, j = 1, 2, 3$

A table with the numbers of cells of the closures of the strata in each dimension is given in Appendix A.

Proof. We show that that the closures $\overline{\mathcal{X}_n}$ consist of the asserted cells. That they are subcomplexes is then a consequence of being closed and of being a union of cells. Since $\overline{\mathcal{X}}_5 \subseteq \cdots \subseteq \overline{\mathcal{X}}_2$, it is convenient to go from $n = 5$ downwards. Then for each n we only have to list cells that are not yet contained in $\overline{\mathcal{X}}_{n+1}$. The subsets $\overline{\mathcal{X}}_n \subseteq \mathcal{X}$ are determined by means of Lemma 7.2 as a union over contributions of the kind $\varphi(\mathcal{A}_i^p \times \mathcal{A}_j^q \times V)$, where V stands for V_τ^0, V_{ij}^1 or V_{ij}^2. Since on passing to $T\backslash\mathrm{U}(3)/T$, the subsets V_τ^0, V_{ij}^1 and V_{ij}^2 pass to the cells K_τ^0, K_{ij}^1 and K_{ij}^2, respectively, we have

$$\varphi(\mathcal{A}_i^p \times \mathcal{A}_j^q \times V) = \pi_{ij}^{pq}(\mathcal{A}_i^p \times \mathcal{A}_j^q \times C)\,,$$

where C is a cell of the double quotient $D(\mathcal{A}_i^p, \mathcal{A}_j^q)$ which is obtained from the cell K_τ^0, K_{ij}^1 or K_{ij}^2 to which V projects in $T\backslash\mathrm{U}(3)/T$ by application of the appropriate factorization map (3.2). We will explain this for $\overline{\mathcal{X}_5}$ and $\overline{\mathcal{X}_4}$ and leave the rest to the reader. For $\overline{\mathcal{X}}_5$, the lemma yields $\overline{\mathcal{X}}_5 = \bigcup_{i,j=1}^3 \varphi\big(\mathcal{A}_i^0 \times \mathcal{A}_j^0 \times \mathrm{U}(3)\big)$. Here, the double quotient is trivial, hence this coincides with

$$\bigcup_{i,j=1}^{3} \pi_{ij}^{00}\big(\mathcal{A}_i^0 \times \mathcal{A}_j^0 \times \mathrm{U}(3)\big) = \bigcup_{i,j=1}^{3} \pi_{ij}^{00}\big(C_{ij}^{000}\big)\,.$$

For $\overline{\mathcal{X}}_4$, from the lemma we read off

$$\overline{\mathcal{X}}_4 \;=\; \bigcup\nolimits_{i,j=1}^{3} \;\Big(\; \varphi\big(\mathcal{A}_i^1 \times \mathcal{A}_j^0 \times \mathrm{U}(3)\big) \;\cup\; \varphi\big(\mathcal{A}_i^0 \times \mathcal{A}_j^1 \times \mathrm{U}(3)\big) \;\cup\; \varphi\big(\mathcal{A}_i^1 \times \mathcal{A}_j^1 \times V_{ij}^1\big) \;\Big)\,.$$

For the first two terms, the double quotient is again trivial, hence these terms yield $\pi_{ij}^{10}(C_{ij}^{100})$ and $\pi_{ij}^{01}(C_{ij}^{010})$. For the third term, the double quotient is $U_i\backslash\mathrm{U}(3)/U_j$ and the cell to which V_{ij}^1 projects is therefore

$$\mu_{ij}^{12} \circ \lambda_i^{12}(K_{ij}^1) = [1]\,.$$

Thus, the third term yields $\pi_{ij}^{11}\,\big(C_{ij1}^{110}\big)$. □

Remark 7. In a previous work [6] it was shown that $\overline{\mathcal{X}}_4$ can be identified with $T_1 \times T_1$, i.e., with a 2-torus, where the discrete subset $\mathbb{Z}_3 \times \mathbb{Z}_3$ corresponds to $\mathcal{X}_5$. Indeed, reading off from the boundary operator how the cells of $\overline{\mathcal{X}}_2$ are attached to one another one can easily see that they add up to a 2-torus, see Figure 3.

8 Homology and cohomology groups

Let us start with recalling some basic facts. The homology groups $H_k(K)$ of an N-dimensional cell complex K are given by

$$H_k(K) := \ker \partial_k / \mathrm{im}\, \partial_{k+1}\,,$$

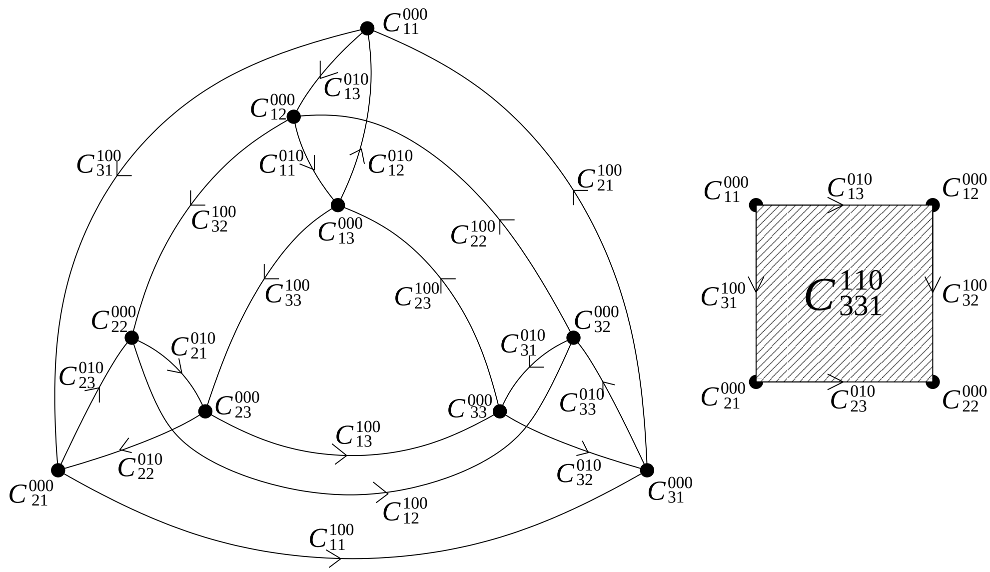

Figure 3: The subcomplex $\overline{\mathcal{X}}_2$. The 2-cells C^{110}_{ij1} are not labelled. As an example, C^{110}_{331} is pictured to the right.

where ∂_k is the boundary homomorphism in dimension k of the chain complex

$$0 \longrightarrow C_N(K) \xrightarrow{\partial_N} C_{N-1}(K) \xrightarrow{\partial_{N-1}} \cdots \xrightarrow{\partial_1} C_0(K) \longrightarrow 0$$

made up by the free abelian groups $C_k(K)$ based on the k-dimensional cells of K. The homology groups $H_k(K,A)$ of K relative to the subcomplex A are given by

$$H_k(K,A) := \ker \tilde{\partial}_k / \mathrm{im}\, \tilde{\partial}_{k+1}\,,$$

where $\tilde{\partial}_k$ is the boundary map in dimension k of the chain complex

$$0 \longrightarrow C_N(K,A) \xrightarrow{\tilde{\partial}_N} C_{N-1}(K,A) \xrightarrow{\tilde{\partial}_{N-1}} \cdots \xrightarrow{\tilde{\partial}_1} C_0(K,A) \longrightarrow 0\,. \tag{8.1}$$

Here $C_k(K,A) = C_k(K)/C_k(A)$ can be identified with the free abelian group based on the k-cells of K not in A and $\tilde{\partial}_k$ can be identified with ∂_k composed with projection to $C_{k-1}(K,A)$. The boundary homomorphisms ∂_k resp. $\tilde{\partial}_k$ can be represented by matrices D_k by numbering the cells in each dimension in an arbitrary way and defining the entry $(D_k)_{ij}$ to be the coefficient with which the j-th $(k-1)$-cell contributes to the image of the i-th k-cell under ∂_k resp. $\tilde{\partial}_k$ (the 'incidence number' of these cells). By construction, the matrices D_k have integer entries and obey $D_k D_{k+1} = 0$. The problem of computing the homology groups is thus reformulated as the problem of computing $\ker D_k / \mathrm{im}\, D_{k+1}$. To solve this, we will apply the following algorithm. Recall that a finitely generated free abelian group G is isomorphic to $\mathbb{Z}^n$. Suppose that $H \subset G$ is a subgroup. Then there exists a basis $e_1, \ldots, e_n$ of G and nonzero integers $q_1, \cdots, q_m$, $m \leqslant n$, such that q_i divides q_{i+1} and $q_1 e_1, \ldots, q_m e_m$ is a basis for H. In particular, H is free abelian of rank m and

$$G/H \simeq \mathbb{Z}^{n-m} \oplus \mathbb{Z}_{q_1} \oplus \cdots \oplus \mathbb{Z}_{q_m}. \tag{8.2}$$

In our case, $G = \ker D_k$ and $H = \operatorname{im} D_{k+1}$. In order to find the numbers m and q_i we can proceed as follows. There exist unimodular matrices U_k and V_k such that $S_k := U_k D_k V_k$ is of the so-called Smith normal form [22]. I.e.,
(i) $(S_k)_{ij} = 0$ for $i \neq j$ (we will sloppily say that S_k is diagonal, although it may not be square)
(ii) $(S_k)_{ii} > 0$ and $(S_k)_{ii}$ divides $S_{i+1\ i+1}$ for $i = 1, \ldots, r$,
(iii) $(S_k)_{ii} = 0$ for $i = r+1, \ldots, n$.
We remark that (S_k) is uniquely determined by D_k, whereas U_k and V_k are not. The columns of the matrix V_k form a basis of the domain $\mathbb{Z}^{n_k}$ of D_k which has the property that the last $n_k - r$ elements $v_{r+1}, \ldots, v_{n_k}$ span $\ker D_k$. Therefore, $\ker D_k \cong \mathbb{Z}^{n_k - r}$. Recall that the dimension $n_k - r$ can be read off from S_k as the number of zero columns. The submatrix

$$P_k^0 := \left\{(V_k^{-1})^i_j\right\}_{j=1\ldots n_k}^{i=r+1\ldots n_k}$$

of V_k^{-1} yields the projection $\mathbb{Z}^{n_k} \to \ker D_k$ which corresponds to the decomposition defined by the basis v_i. We define $D_{k+1}^0 := P_k^0 D_{k+1} : \mathbb{Z}^{n_{k+1}} \to \ker D_k$. As $\operatorname{im} D_{n_k+1} \subseteq \ker D_k$, we have $\operatorname{im} D_{n_k+1} = \operatorname{im} D_{n_k+1}^0$ and hence $H_k = \ker D_k / \operatorname{im} D_{k+1}^0 \cong \mathbb{Z}^{n_k - r} / \operatorname{im} S_{k+1}^0$, where S_{k+1}^0 denotes the Smith normal form of D_{k+1}^0. Since S^0 is diagonal, we can analyze the factorization in each component independently. Every row equal to zero in S^0 corresponds to a generator in $\mathbb{Z}^{n_k - r}$ which does not appear in $\operatorname{im} S^0$ and hence generates a factor $\mathbb{Z}$ in the quotient $\mathbb{Z}^{n_k - r}/\operatorname{im} S^0$. Each nonzero diagonal entry q contributes a finite cyclic factor $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ to the quotient, where of course the factors with $q = 1$ can be omitted. Summarizing,

$$H_k = \mathbb{Z}^l \oplus_q \mathbb{Z}_q\,,$$

where l is the number of zero rows in S^0 and q runs through the diagonal entries distinct from 0 and 1.
Let us illustrate the procedure with the following example. Consider the group $H_4(\overline{\mathcal{X}}_2)$. Due to Theorem 7.3, the cell complex $\overline{\mathcal{X}}_2$ has 9, 24 and 27 cells in dimension 5, 4 and 3 respectively. Hence, the corresponding part of the chain complex (8.1) reads

$$\mathbb{Z}^9 \xrightarrow{D_5} \mathbb{Z}^{24} \xrightarrow{D_4} \mathbb{Z}^{27}$$

where D_5 and D_4 consist of the incidence numbers given in Theorem 6.1. First we derive the Smith normal form $S_4 = U_4 D_4 V_4$ of D_4. It turns out that for $i = 1, \ldots, 13$, ${S^i}_i = 1$ and all other entries vanish, so there are $24 - 13 = 11$ zero columns. Thus, $\ker D_4 \cong \mathbb{Z}^{11}$ and it is generated by the last 11 columns of the matrix V_4. So the last 11 rows of the matrix V_4^{-1} define the projection $P^0 = \left\{(V^{-1})^i_j\right\}_{j=1,\ldots,24}^{i=14,\ldots,24}$ onto $\ker D_4$. Then $D_5^0 = P^0 D_5$ is obtained by expressing D_5 in the basis defined by V and neglecting the first 13 zero rows. The Smith normal form of D_5^0 is

$$S_5^0 = \begin{bmatrix} \mathbb{1}_8 & \mathbf{0}_8 \\ \mathbf{0}_8^T & 3 \\ \mathbf{0}_8^T & 0 \\ \mathbf{0}_8^T & 0 \end{bmatrix},$$

where $\mathbb{1}_8$ and $\mathbf{0}_8$ denote the 8-dimensional unit matrix and the 8-dimensional zero vector, rescpectively. Thus, we read off $H_4(\overline{\mathcal{X}}_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$.
The computation of homology groups can be fully automatized by programming the above algorithm on the computer. The program is written in *Maple 8* and uses built-in routines for computing Smith normal forms. This way, we obtain

Theorem 8.1. *The homology groups of $\mathcal{X}$ and of the subcomplexes $\overline{\mathcal{X}}_i$ are*

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$\mathcal{X}$	$\mathbb{Z}$	0	0	0	0	0	0	0	$\mathbb{Z}$
$\overline{\mathcal{X}}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_6$	0	0	0	0
$\overline{\mathcal{X}}_3$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0
$\overline{\mathcal{X}}_4$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0	0	0	0	0	0
$\overline{\mathcal{X}}_5$	$\mathbb{Z}^9$	0	0	0	0	0	0	0	0

In particular, $\mathcal{X}$ has the homology of an 8-sphere. □

We comment on the last statement of the theorem at the end of this section.
Next, we are going to compute the homology and cohomology groups of the strata. We will use the following facts. Let X be a compact space and let $A \subseteq X$ be a closed subset such that $X \setminus A$ is an orientable n-manifold. Then

$$H_q(X \setminus A) \cong H^{n-q}(X, A; \mathbb{Z}), \qquad H^q(X \setminus A, \mathbb{Z}) \cong H_{n-q}(X, A), \tag{8.3}$$

where $H^k(X, A; \mathbb{Z})$ denotes the k-th cohomology group of X relative to A with coefficient group $\mathbb{Z}$, see [9]. We wish to apply (8.3) to $X = \overline{\mathcal{X}}_i$ and $A = \overline{\mathcal{X}}_{i+1}$ (thus $X \setminus A = \mathcal{X}_i$). First, applying the above algorithm to the cell complexes $\overline{\mathcal{X}}_i \setminus \overline{\mathcal{X}}_{i+1}$ we compute the relative homology groups:

	H_0	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8
$(\mathcal{X}, \overline{\mathcal{X}}_2)$	0	0	0	0	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$	0	0	$\mathbb{Z}$
$(\overline{\mathcal{X}}_2, \overline{\mathcal{X}}_3)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0	0	0
$(\overline{\mathcal{X}}_3, \overline{\mathcal{X}}_4)$	0	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_6$	0	$\mathbb{Z}$	0	0	0	0
$(\overline{\mathcal{X}}_4, \overline{\mathcal{X}}_5)$	0	$\mathbb{Z}^{10}$	$\mathbb{Z}$	0	0	0	0	0	0

(8.4)

Second, we have to make sure that $\mathcal{X}_i$ is orientable. We will apply the following lemma.

Lemma 8.2. *Let X be a cell complex of dimension n and let A be a subcomplex, $A \neq X$, such that $X \setminus A$ is a connected n-manifold. If $H_n(X, A) \neq 0$ then $X \setminus A$ is orientable.*

Proof. Let C_i denote the n-cells of X. Let $C = \sum_i k_i C_i$ be a chain in X which is a cycle modulo A and which generates $H_n(X, A)$. There exists i_0 such that $k_{i_0} \neq 0$ and $\dot{C}_{i_0} \cap A = \emptyset$. Choose $x \in \dot{C}_{i_0}$. Let $i : (X, A) \to (X, X \setminus \{x\})$ denote the natural injection. We will show that $i_* C \neq 0$. Then the assertion follows from Lemma 3 in [21, §IV.3.3]. Let $f_{i_0} : \sigma^n \to X$ denote the characteristic map of C_{i_0}. Choose a subdivision $\{\sigma^n_k\}$ of σ^n such that the simplex σ^n_0 contains $f^{-1}_{i_0}(x)$ in its interior. By shrinking σ^n to σ^n_0 and composing with $f_{i_0}|_{\sigma^n_0}$ we obtain a singular chain $C'_0 = (\sigma^n, f'_0)$ in X whose support contains x and which is embedded homeomorphically. The remaining simplices σ^n_k of the subdivision yield singular chains $C'_k = (\sigma^n_k, f_{i_0}|_{\sigma^n_k})$, $k \neq 0$. By construction, $\sum_k C'_k$ is homologous in X to C_{i_0}, viewed as a singular chain. Now consider $H_n(X, X \setminus \{x\})$. As C'_0 is embedded homeomorphically, $i_* C'_0$ is a generator of $H_n(X, X \setminus \{x\})$, hence $i_* C'_0 \neq 0$. Since $x \in X \setminus A$ and $X \setminus A$ is an n-manifold, $H_n(X, X \setminus \{x\})$ is free. Hence, $k_{i_0} i_* C'_0 \neq 0$. We claim that $k_{i_0} i_* C'_0$ and $i_* C$ are homologous in $(X, X \setminus \{x\})$: the singular chain $C = \sum_{i \neq i_0} k_i C_i + k_{i_0} C_{i_0}$ is homologous in X to $\sum_{i \neq i_0} k_i C_i + k_{i_0} \sum_{k \neq 0} C'_k + k_{i_0} C'_0$. Since all C_i, $i \neq i_0$, and all C'_k, $k \neq 0$, are contained in $X \setminus \{x\}$, the assertion follows. This proves the lemma. □

We check that $X = \overline{\mathcal{X}}_i$ and $A = \overline{\mathcal{X}}_{i+1}$ obey the assumptions of the lemma. For the assumption on the homology group this follows from Table (8.4). For the assumption on connectedness, we have

Lemma 8.3. *The strata $\mathcal{X}_1, \dots, \mathcal{X}_4$ are connected.*

Proof. For $\mathcal{X}_4$, the assertion is obvious. For $\mathcal{X}_1$, we use that it is obtained from the connected cell complex $\mathcal{X}$ by removing the subcomplex $\overline{\mathcal{X}}_2$ which has codimension 3. The same argument applies to $\mathcal{X}_3 = \overline{\mathcal{X}}_3 \setminus \overline{\mathcal{X}}_4$, because $\overline{\mathcal{X}}_4$ has codimension 2 in $\overline{\mathcal{X}}_3$ and from the homology groups we read off that $\overline{\mathcal{X}}_3$ is connected. For $\mathcal{X}_2 = \overline{\mathcal{X}}_2 \setminus \overline{\mathcal{X}}_3$, we cannot apply this argument, because the codimension of $\overline{\mathcal{X}}_3$ in $\overline{\mathcal{X}}_2$ is 1. Here, we use that $\overline{\mathcal{X}}_2$ is made up by the nine 5-cells C^{221}_{ij}, $i, j = 1, 2, 3$. Using the boundary formulae in Theorem 6.1 we check

$$C^{221}_{ij} \cap C^{221}_{i\, j_\pm} \supseteq C^{211}_{j_\mp\, i}\,, \qquad C^{221}_{ij} \cap C^{221}_{i_\pm\, j} \supseteq C^{121}_{i_\mp\, j}\,.$$

Hence, starting from inside C^{221}_{11} one can reach any of the 5-cells C^{221}_{ij} on a path inside the union of the interiors of these 5-cells and the interiors of the 4-cells C^{211}_{ij} and C^{121}_{ij}, $i, j = 1, 2, 3$. Since the latter do not belong to $\overline{\mathcal{X}}_3$, this shows that $\mathcal{X}_2$ is connected, too. □

Now, application of (8.3) yields

Theorem 8.4. *The integer cohomology groups of the strata are*

	H^0	H^1	H^2	H^3
$\mathcal{X}_1$	$\mathbb{Z}$	0	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$
$\mathcal{X}_2$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0
$\mathcal{X}_3$	$\mathbb{Z}$	0	$\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_6$	0
$\mathcal{X}_4$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	0	0

All the remaining cohomology groups are trivial. $\square$

To compute the homology groups we need the relative cohomology groups $H^k(\overline{\mathcal{X}}_i, \overline{\mathcal{X}}_{i+1}; \mathbb{Z})$. According to the Universal Coefficient Theorem, see e.g. [5, Cor. 7.3],

$$H^k(\overline{\mathcal{X}}_i, \overline{\mathcal{X}}_{i+1}; \mathbb{Z}) \cong F_k \oplus T_{k-1},$$

where F_k and T_k are the free and the torsion part, respectively, of $H_k(\overline{\mathcal{X}}_i, \overline{\mathcal{X}}_{i+1})$. We obtain

	H^0	H^1	H^2	H^3	H^4	H^5	H^6	H^7	H^8
$(\mathcal{X}, \overline{\mathcal{X}}_2)$	0	0	0	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_3$	0	$\mathbb{Z}$
$(\overline{\mathcal{X}}_2, \overline{\mathcal{X}}_3)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0	0	0
$(\overline{\mathcal{X}}_3, \overline{\mathcal{X}}_4)$	0	0	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z}$	0	0	0	0
$(\overline{\mathcal{X}}_4, \overline{\mathcal{X}}_5)$	0	$\mathbb{Z}^{10}$	$\mathbb{Z}$	0	0	0	0	0	0

Then (8.3) yields

Theorem 8.5. *The homology groups of the strata $\mathcal{X}_1, \ldots, \mathcal{X}_4$ are*

	H_0	H_1	H_2	H_3
$\mathcal{X}_1$	$\mathbb{Z}$	0	$\mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}$
$\mathcal{X}_2$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{Z}$	0
$\mathcal{X}_3$	$\mathbb{Z}$	$\mathbb{Z}_6$	$\mathbb{Z} \oplus \mathbb{Z}$	0
$\mathcal{X}_4$	$\mathbb{Z}$	$\mathbb{Z}^{10}$	0	0

All other homology groups are trivial. $\square$

Next, we list those homotopy groups of the skeleta and the closures of the strata which follow by general theorems from our results above. Computation of further homotopy groups remains a future task.

Corollary 8.6.
(i) *For $n \geq 2$, the n-skeleton $\mathcal{X}^n$ is $(n-1)$-connected. Moreover,*

$$\pi_2(\mathcal{X}^2) = \mathbb{Z}^{14},\ \pi_3(\mathcal{X}^3) = \mathbb{Z}^{13},\ \pi_4(\mathcal{X}^4) = \mathbb{Z}^{11},\ \pi_5(\mathcal{X}^5) = \mathbb{Z}^4,\ \pi_6(\mathcal{X}^6) = \mathbb{Z}^5,\ \pi_7(\mathcal{X}^7) = \mathbb{Z}\,.$$

(ii) *The 1-skeleton $\mathcal{X}^1$ and the stratum $\mathcal{X}_4$ are homotopy-equivalent to a bouquet of ten 1-spheres. In particular, their fundamental group is the free group on 10 generators and the other homotopy groups are trivial.*

(iii) *The principal stratum $\mathcal{X}_1$ has $\pi_1(\mathcal{X}_1) = 0$ and $\pi_2(\mathcal{X}_1) = \mathbb{Z}_3$.*

(iv) *The lower homotopy groups of the closures of the strata are*

$$\begin{aligned}
\pi_k(\overline{\mathcal{X}}_2) &= \begin{cases} 0 & \mid k = 0, \ldots, 3\,, \\ \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 & \mid k = 4\,, \end{cases} \\
\pi_k(\overline{\mathcal{X}}_3) &= \begin{cases} 0 & \mid k = 0, 1\,, \\ \mathbb{Z} & \mid k = 2\,, \end{cases} \\
\pi_k(\overline{\mathcal{X}}_4) &= \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \mid k = 2\,, \\ 0 & \mid \textit{otherwise.} \end{cases}
\end{aligned}$$

Proof. (i) Let $n \leq 7$. Since $\mathcal{X}$ is obtained from the n-skeleton $\mathcal{X}^n$ by attaching cells of dimension $n+1$ and higher, $\pi_k(\mathcal{X}^n) = \pi_k(\mathcal{X}) = 0$ for all $k < n$. This is a consequence of the Whitehead theorem, see [5, Prop. VII.11.6]. Then, for $n \geq 2$, the Hurewicz theorem implies $\pi_n(\mathcal{X}^n) \cong H_n(\mathcal{X}^n)$. The latter is just the kernel of the boundary map in dimension n, hence it is a free Abelian group. Its rank can be determined as follows. Since $H_n(\mathcal{X}) = 0$, the rank coincides with the rank of the image of the boundary map in dimension $n+1$, which is given by the number of $n+1$-cells minus the rank of the kernel of the boundary map in dimension $n+1$. Iterating this argument, we obtain that the rank of $H_n(\mathcal{X}^n)$ is given by the alternating sum of the numbers of cells of dimensions $n+1$ to 7 plus/minus 1 for the rank of $H_8(\mathcal{X})$.

(ii) $\mathcal{X}^1$ is a graph with 9 vertices and 18 edges, hence it is homotopy-equivalent to a bouquet of $(1 - 9 + 18)$ spheres of dimension 1. $\mathcal{X}_4$ is a 9-punctured 2-torus, hence it is homotopy-equivalent to a bouquet of 10 spheres of dimension 1.

(iii) $\mathcal{X}_1$ is obtained from $\mathcal{X}$ by removing $\overline{\mathcal{X}}_2$ which consists of cells of codimension 3. Hence $\pi_i(\mathcal{X}_1) = \pi_i(\mathcal{X})$ for $i \leq 1$. Then the Hurewicz theorem implies $\pi_2(\mathcal{X}_1) = H_2(\mathcal{X}_1) = \mathbb{Z}_3$.

(iv) For $\overline{\mathcal{X}}_4$, the result is obvious because $\overline{\mathcal{X}}_4 \cong \mathrm{S}^1 \times \mathrm{S}^1$, see Figure 3 and the corresponding explanation in the text. For $\overline{\mathcal{X}}_3$ we observe that it is obtained from the 2-skeleton $\mathcal{X}^2$ by attaching cells of dimension ≥ 3. Hence, by [5, Prop. VII.11.6] again, it has the same π_0 and π_1 as $\mathcal{X}^2$. Both groups vanish due to (i). Then the Hurewicz theorem implies $\pi_2(\overline{\mathcal{X}}_3) \cong H_2(\overline{\mathcal{X}}_3) = \mathbb{Z}$. For $\overline{\mathcal{X}}_2$, the argument is completely analogous. □

Let us conclude with some remarks. Besides the algebraic characterizations derived above, we have seen that the lowest dimensional strata $\mathcal{X}_5$ and $\mathcal{X}_4$ can be identified with standard topological spaces: $\mathcal{X}_5$ is a discrete space consisting of nine elements. $\overline{\mathcal{X}}_4$ is homeomorphic to a 2-torus and hence $\mathcal{X}_4$ is diffeomorphic to a 9-punctured 2-torus. Thus, the question arises whether the other strata and their closures as well as the entire reduced configuration space $\mathcal{X}$ can be expressed in terms of standard topological building blocks like spheres or projective spaces, too. E.g., we have found that $\mathcal{X}$ has the homology groups of an 8-sphere. Since it is a cell complex and simply connected, it is then homotopy-equivalent to an 8-sphere. In fact, it seems likely that $\mathcal{X}$ is in fact homeomorphic to an 8-sphere. However, at the moment we are not able to prove or disprove this. Note that $\mathcal{X}$ is not a differentiable

manifold, hence the famous Poincaré conjecture does not apply here. The crucial point is to check whether $\mathcal{X}$ has the so-called disjoint disks property [8].
As another example, we have found that $\overline{\mathcal{X}}_3$ has the homology groups of the complex projective space $\mathbb{C}P^2$. Contrary to the case of a sphere, this does not even imply that the two spaces are homotopy-equivalent; a counterexample is provided by the bouquet of a 2 and a 4-sphere. Nevertheless, we attempted to clarify whether the two spaces are homeomorphic but failed until now. Let us outline the strategy we followed. If one contracts a generator of $H_2(\overline{\mathcal{X}}_3)$ which is a subcomplex homeomorphic to a 2-sphere to a point one arrives at a simply connected quotient cell complex which has the homology groups of the 4-sphere and hence is homotopy-equivalent to the latter. If one could prove that the quotient cell complex is homeomorphic to a 4-sphere then one would find that $\overline{\mathcal{X}}_3$ is obtained by attaching a 4-cell to a 2-sphere. If further one could show that the attaching map is the Hopf map then $\overline{\mathcal{X}}_3$ would be homeomorphic to $\mathbb{C}P^2$, indeed.

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A Table of number of cells

dim	$\mathcal{X}_5$	$\overline{\mathcal{X}}_4$	$\overline{\mathcal{X}}_3$	$\overline{\mathcal{X}}_2$	$\mathcal{X}$
0	9	9	9	9	9
1	–	18	18	18	18
2	–	9	24	24	24
3	–	–	18	27	27
4	–	–	6	24	24
5	–	–	–	9	15
6	–	–	–	–	9
7	–	–	–	–	6
8	–	–	–	–	2
total	9	36	75	111	134

B Relation between the cells of $T\backslash\mathrm{U}(3)/T$ and Schubert cells of $\mathrm{U}(3)/T$

We relate the cells of $T\backslash\mathrm{U}(3)/T$ constructed above with the Bruhat cells in the complexification $\mathrm{GL}(3,\mathbb{C})$ of $\mathrm{U}(3)$ and the Schubert cells in the flag manifold $\mathrm{U}(3)/T$, respectively. Let $B \subseteq \mathrm{GL}(3,\mathbb{C})$ denote the subgroup of upper triangular matrices and define $B_\tau := \tau B \tau^{-1}$, $\tau \in S_3$. The subgroups B_τ are Borel subgroups of $\mathrm{GL}(3,\mathbb{C})$ associated with the Cartan subalgebra of $\mathrm{gl}(3,\mathbb{C})$ of diagonal matrices and a certain choice of base for the corresponding root system. The Bruhat cells of $\mathrm{GL}(3,\mathbb{C})$ relative to B_τ are the subsets $B_\tau\tau' B_\tau$, $\tau' \in S_3$. These subsets provide a disjoint decomposition

$$\mathrm{GL}(3,\mathbb{C}) = \bigcup_{\tau' \in S_3} B_\tau \tau' B_\tau \,, \tag{2.1}$$

the Bruhat decomposition. By intersection with the maximal compact subgroup $\mathrm{U}(3)$, the Bruhat decomposition induces a decomposition of $\mathrm{U}(3)$:

$$\mathrm{U}(3) = \bigcup_{\tau' \in S_3} (B_\tau \tau' B_\tau) \cap \mathrm{U}(3) \,.$$

Explicitly, the intersections are given by

	τ'					
τ	τ_0	τ_1	τ_3	τ_4	τ_5	τ_2
τ_0	V^0_τ	V^1_{11}	V^1_{33}	$V^2_{13} \setminus V^1_{11} \setminus V^1_{33}$	$V^2_{31} \setminus V^1_{11} \setminus V^1_{33}$	$\mathrm{U}(3) \setminus V^2_{13} \setminus V^2_{31}$
τ_1	V^0_τ	V^1_{11}	V^1_{22}	$V^2_{12} \setminus V^1_{11} \setminus V^1_{22}$	$V^2_{21} \setminus V^1_{11} \setminus V^1_{22}$	$\mathrm{U}(3) \setminus V^2_{12} \setminus V^2_{21}$
τ_2	V^0_τ	V^1_{33}	V^1_{11}	$V^2_{31} \setminus V^1_{11} \setminus V^1_{33}$	$V^2_{13} \setminus V^1_{11} \setminus V^1_{33}$	$\mathrm{U}(3) \setminus V^2_{13} \setminus V^2_{31}$
τ_3	V^0_τ	V^1_{22}	V^1_{33}	$V^2_{23} \setminus V^1_{22} \setminus V^1_{33}$	$V^2_{32} \setminus V^1_{22} \setminus V^1_{33}$	$\mathrm{U}(3) \setminus V^2_{23} \setminus V^2_{32}$
τ_4	V^0_τ	V^1_{33}	V^1_{22}	$V^2_{32} \setminus V^1_{22} \setminus V^1_{33}$	$V^2_{23} \setminus V^1_{22} \setminus V^1_{33}$	$\mathrm{U}(3) \setminus V^2_{23} \setminus V^2_{32}$
τ_5	V^0_τ	V^1_{22}	V^1_{11}	$V^2_{21} \setminus V^1_{11} \setminus V^1_{22}$	$V^2_{12} \setminus V^1_{11} \setminus V^1_{22}$	$\mathrm{U}(3) \setminus V^2_{12} \setminus V^2_{21}$

The Bruhat cells project to open cells of the flag manifold $\mathrm{U}(3)/T \cong \mathrm{GL}(3,\mathbb{C})/B_\tau$, the Schubert cells associated with B_τ. The dimensions of the Schubert cells are, in the order of the cells as in the table, $0, 2, 2, 4, 4, 6$. According to the table, when further factorizing by left multiplication by T, the 0-dimensional Schubert cell projects to $K^0_{\tau_0}$, the 2-dimensional Schubert cells project to two of the 1-dimensional cells K^1_{11}, K^1_{22} or K^1_{33}, the 4-dimensional Schubert cells project to two of the 2-dimensional cells K^2_{ij}, $i \neq j$, and the 6-dimensional Bruhat cell projects to the 4-dimensional complement in $T\backslash\mathrm{U}(3)/T$ of the two 2-cells. Inspection of the boundaries of the two 2-cells in $T\backslash\mathrm{U}(3)/T$ yields that this complement is contractible. Although it is not a cell, it can still be interpreted as being some 4-disk attached to the two 2-cells under consideration. This way, the Bruhat cells w.r.t. an

arbitrary but fixed Borel subgroup B_τ of $\mathrm{GL}(3,\mathbb{C})$ yield a certain cell decomposition of $T\backslash \mathrm{U}(3)/T$ (with non-canonical 4-cell though). Now consider the Bruhat decompositions w.r.t. all the Borel subgroups B_τ, $\tau \in \mathcal{S}_3$. A common subdivision is given by the subsets

$$B_\tau \sigma B_\rho \,, \qquad \tau, \sigma, \rho \in \mathcal{S}_3$$

(note that the labelling by three permutations contains redundancies), see [10]. Since $B_\tau \sigma B_\rho = \tau B \sigma' B \rho^{-1}$ with $\sigma' = \tau^{-1}\sigma\rho$, by (7.2), on passing to $T\backslash \mathrm{U}(3)/T$, these subsets yield any 0, 1 and 2-cell, as well as the complement of all these cells in $T\backslash \mathrm{U}(3)/T$. Hence, it is this common subdivision to which the cell decomposition constructed above corresponds on the level of Bruhat cells.

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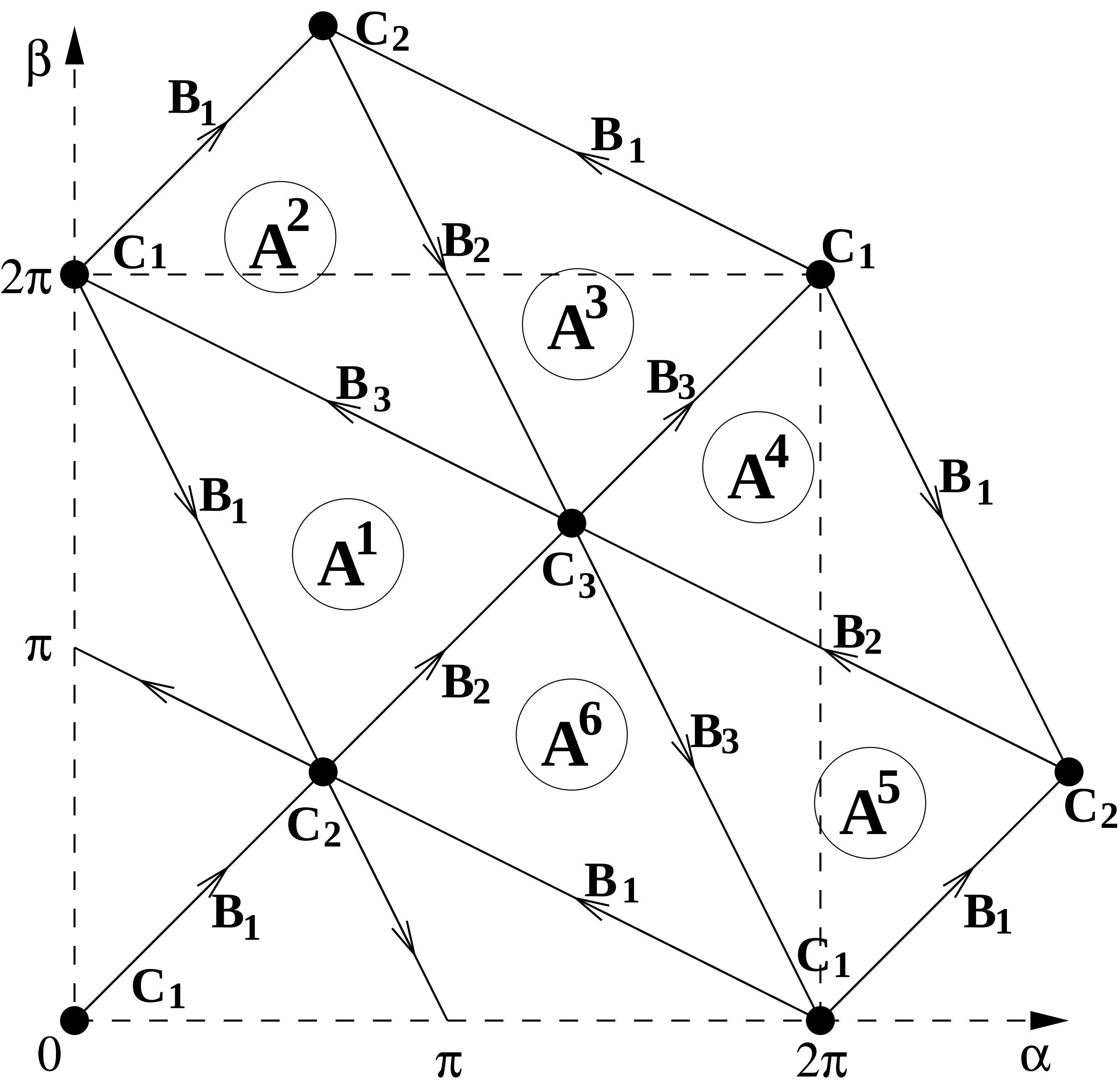

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